An introduction to combinatorial games

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Introduction

Games with:

- 2 players playing alternately;
- perfect information.

Chess

Card games

Othello

Draughts

Tic Tac Toe

Pachisi

Go

Main question:

Who is winning and how?

→ Exact and approximate resolutions

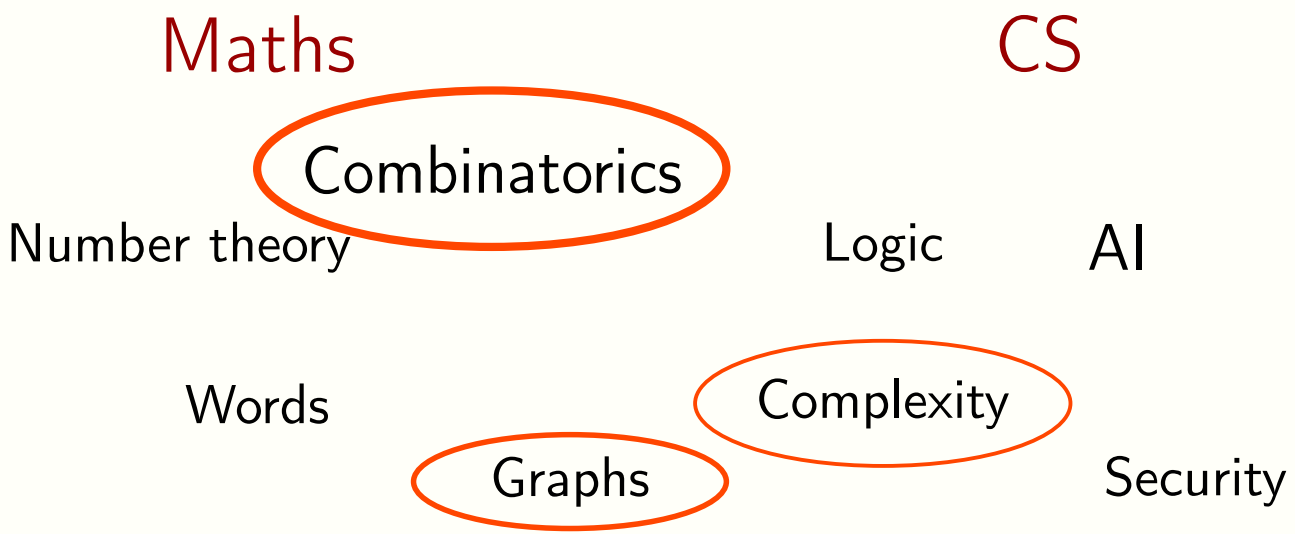
Introduction

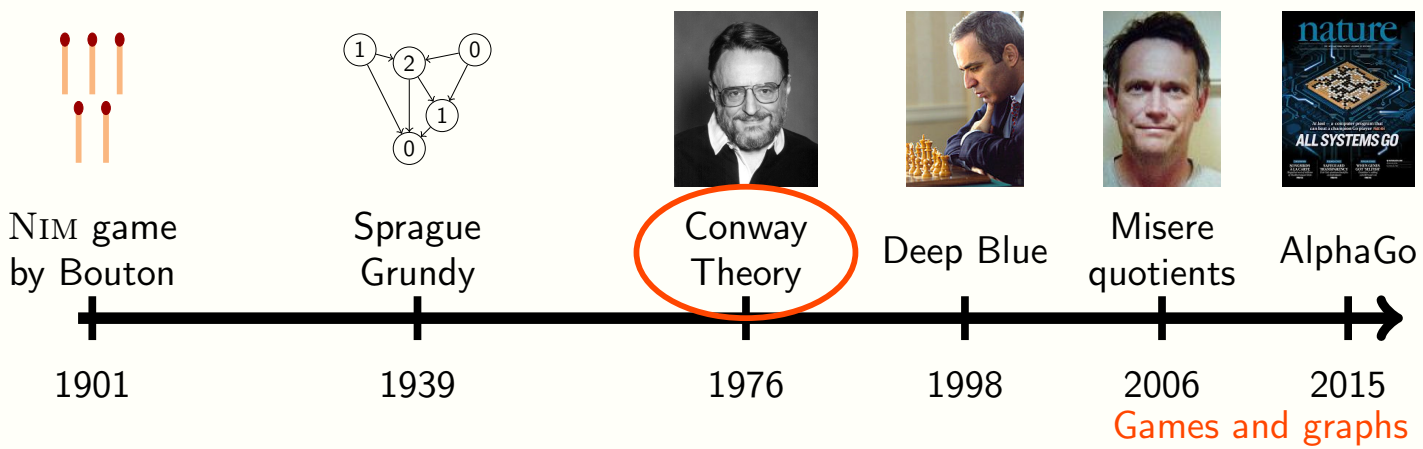
What

Why

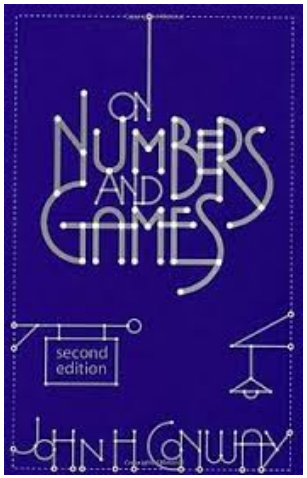
Who

When

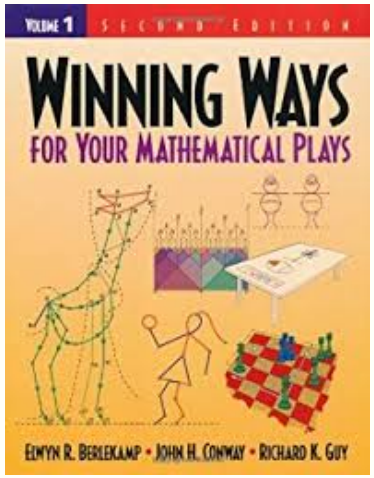




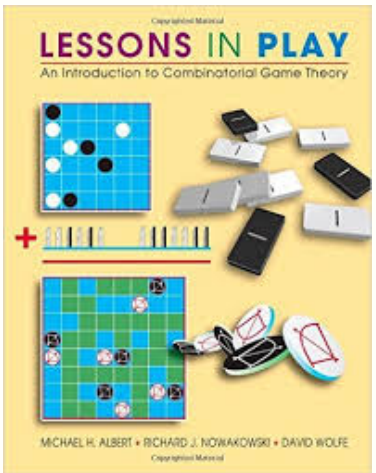
Reference books



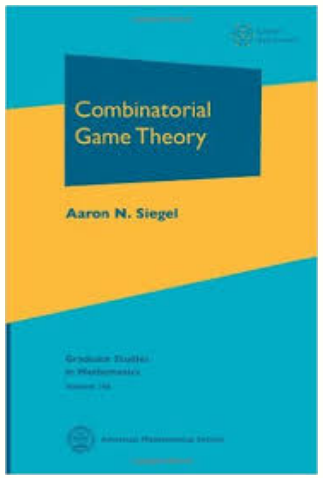
1976



1981



2007



2013

Pure combinatorial games - a definition

Berlekamp, Conway, Guy (Winning Ways, 1981)

- 2 players: Left and Right, that play alternately and cannot pass their turn;
- Perfect information, no chance;
- Finite number of moves, no draw, always a winner;
- Winner determined according to the last move (no scoring)


Chess


Card games


Othello


Draughts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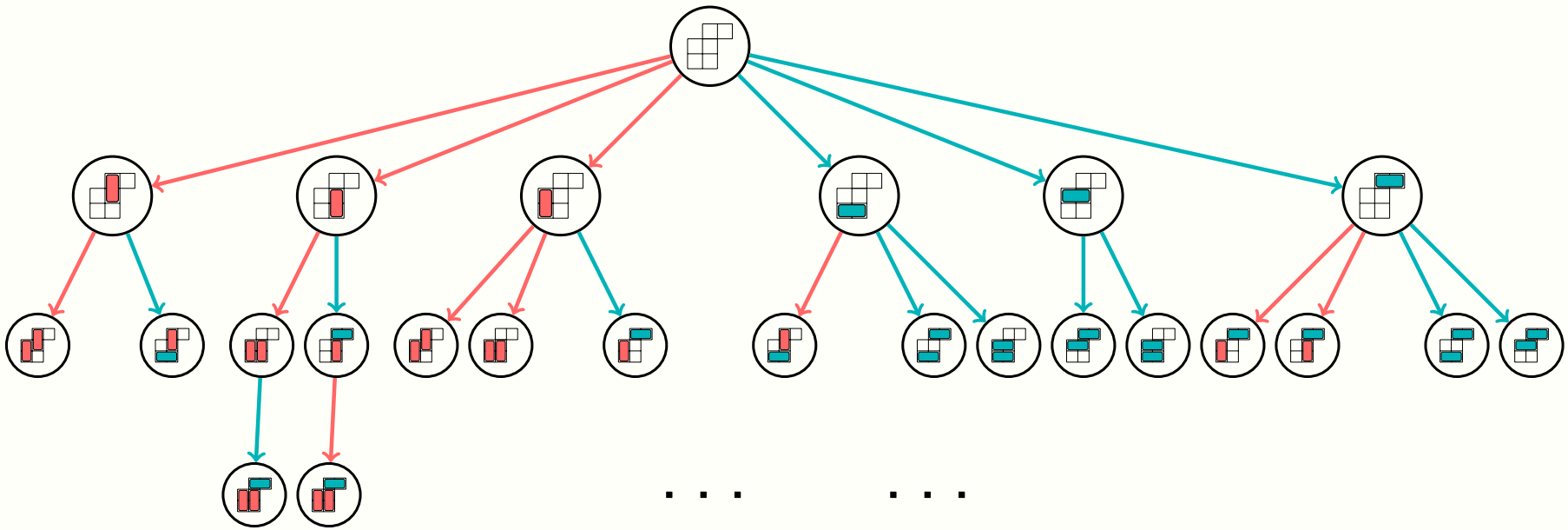
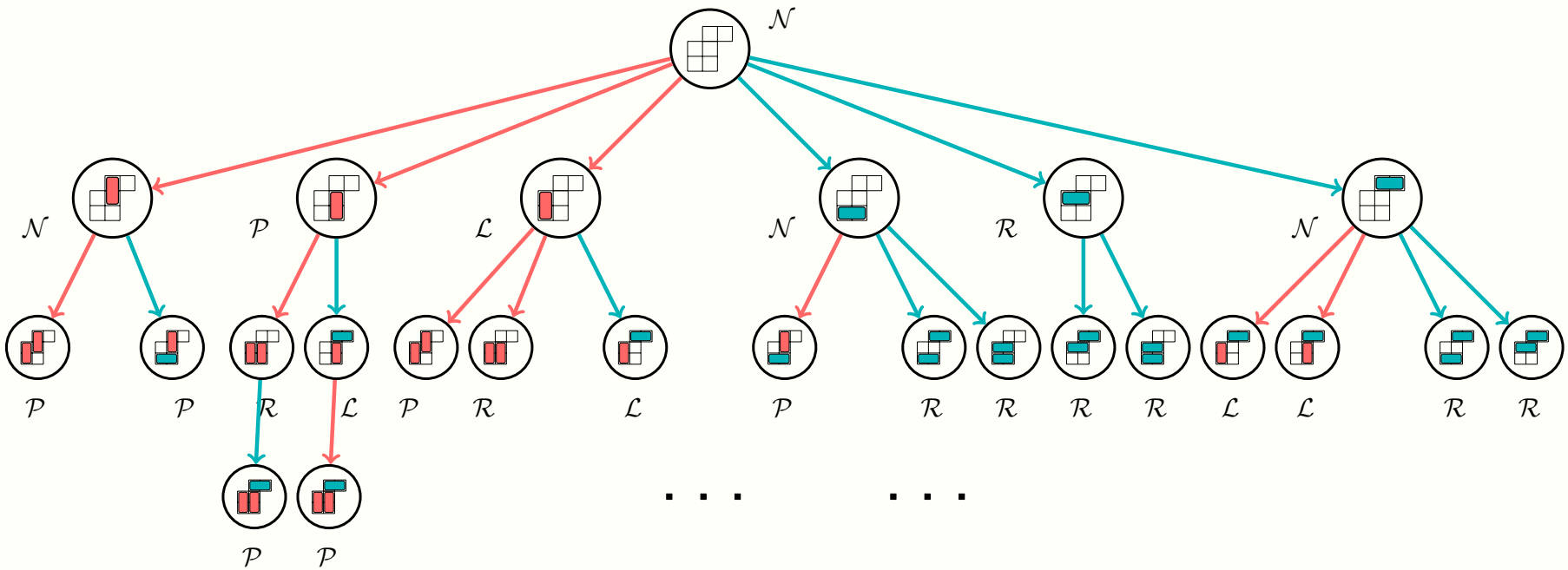
Pachisi


Go

Game tree



Game tree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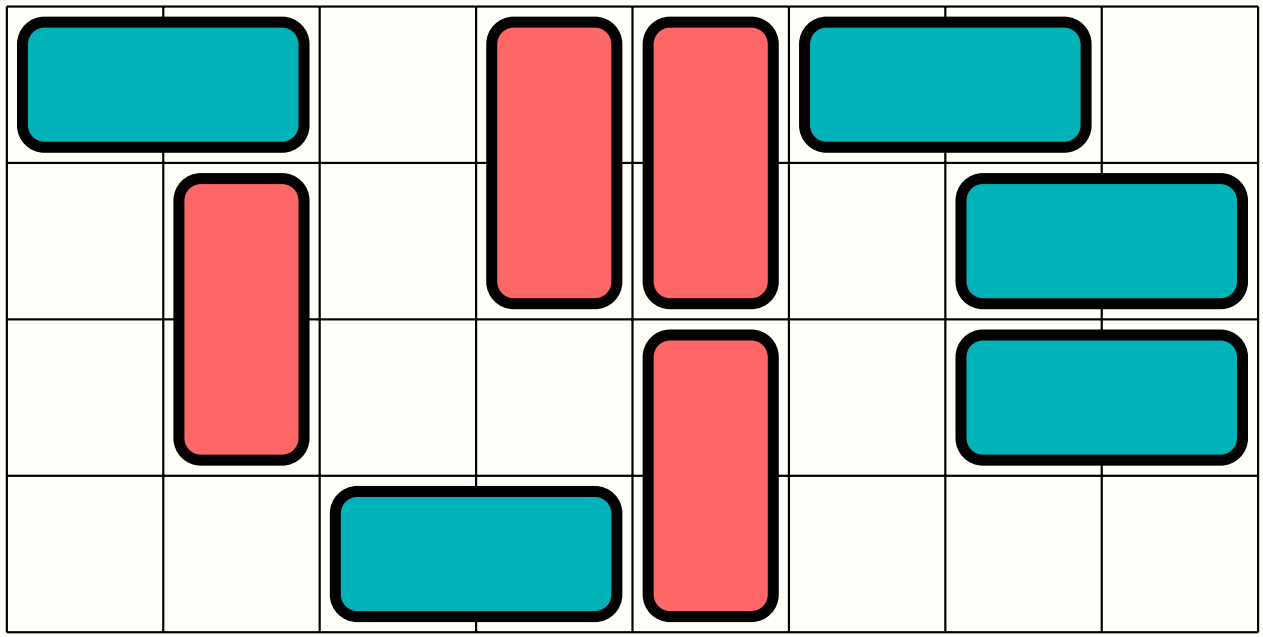
Computing the outcome of DOMINEERING

- Unknown complexity on a $n \times m$ board.
- When n and m are fixed:

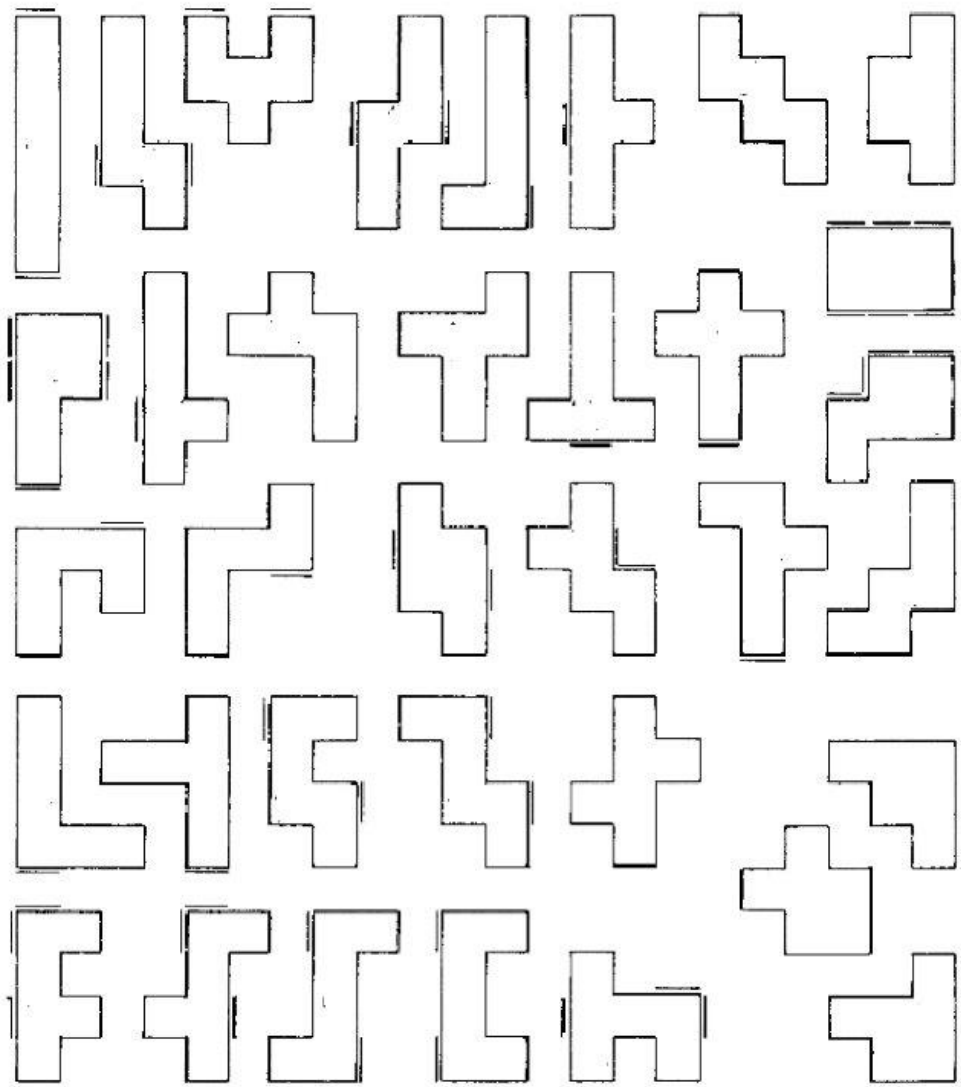
m\n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	2	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
2	V	1	1	H	V	1	1	H	V	1	1	H	2	1	1	H	H	1	1	H	H	H	1	H	H	H	1	H	H
3	V	1	1	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
4	V	V	V	1	V	1	V	H	V	H	V	H	2	H	H	H	H	H	**H**	H	**H**	H	H	H	H	H	H	H	H
5	V	H	V	H	2	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
6	V	1	V	1	V	1	V	H	V	1	1	H	V	**H**		H		1h	1h	H		H	1h	H	1h	H	1h	H	1h
7	V	1	V	H	V	H	1	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
8	V	V	V	V	V	V	V	1	V	**H**	V		V			2h		1h		H				1h					
9	V	H	V	H	V	H	V	H	1	H	1h	H	1h	H	1h	H	1h	H	1h	H	1h	H	H	H	H	H	H	H	H
10	V	1	V	V	V	1	V	**V**	V	**1**			V							2h		1h				1h			
11	V	1	V	H	V	1	V	H	1v		12	H		1h	1h	H		1h	1h	H		H	1h	H		H	1h	H	

[Bullcock's website]

Decomposing DOMINEERING into a sum of games



How to play on a big DOMINEERING game ?



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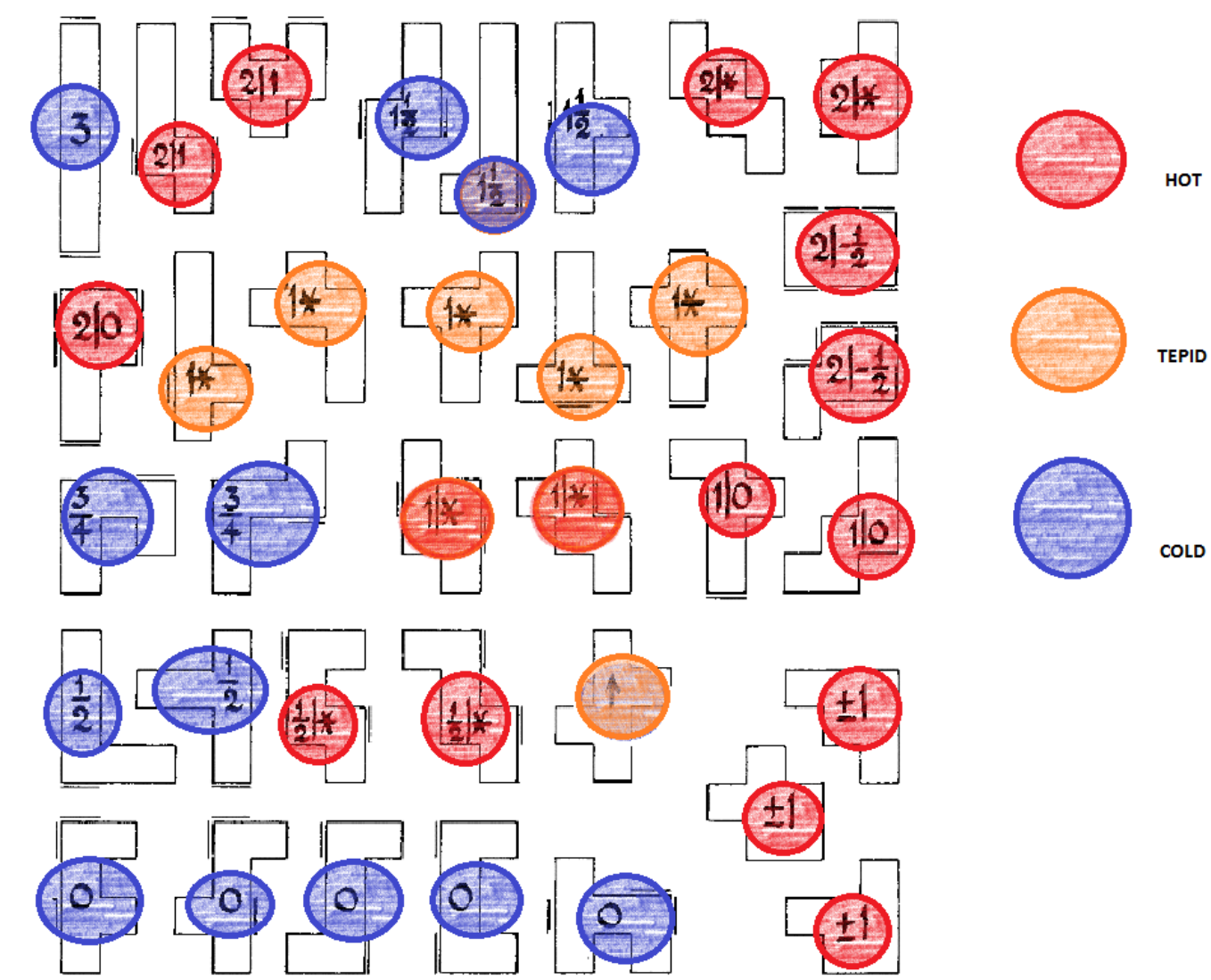
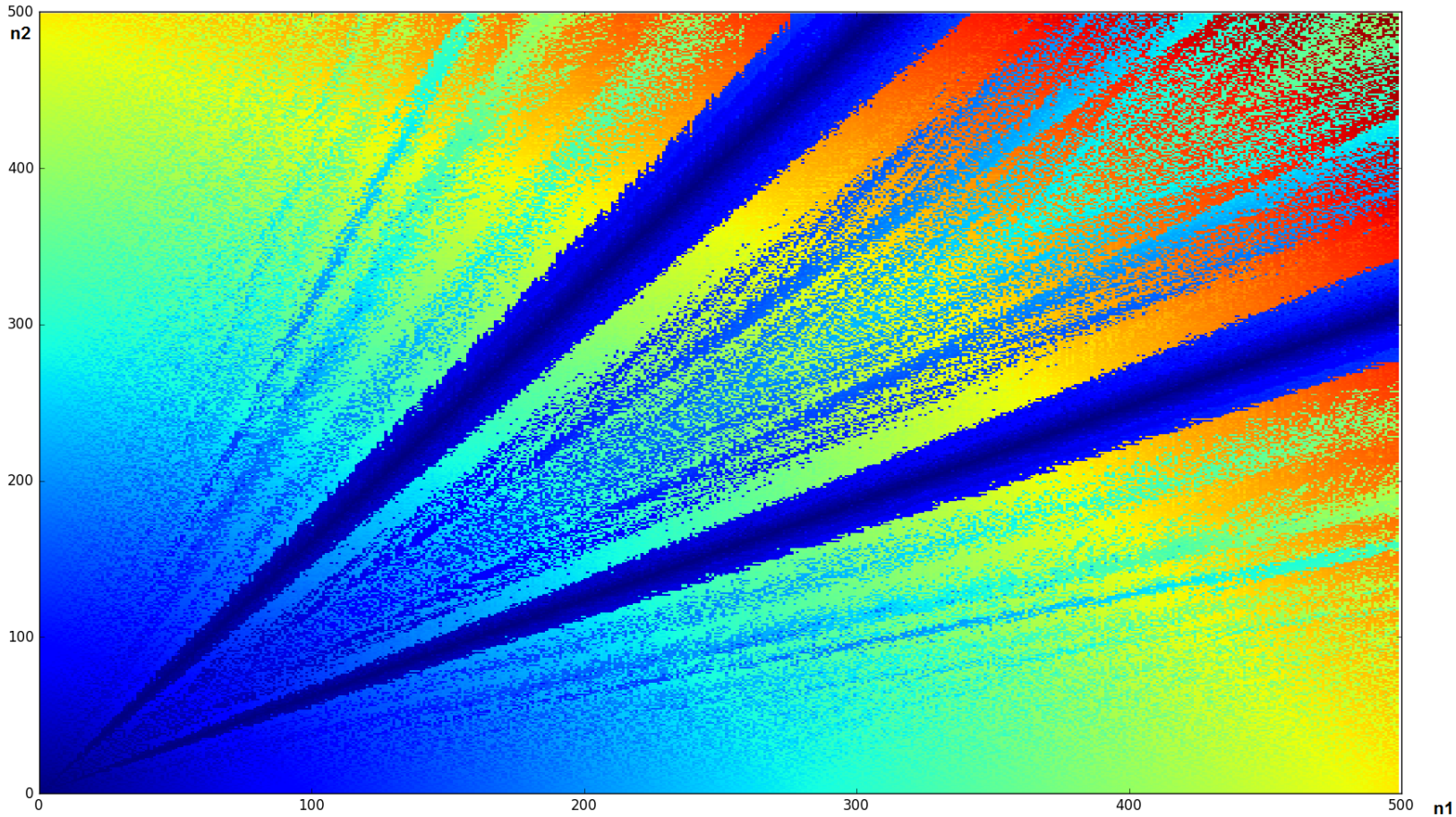


Values of positions of WYTHOFF



The PSPACE class

Definition: a decision problem is PSPACE if it can be solve in polynomial space by a Turing Machine.

The standard PSPACE-complete problem :

QUANTIFIED BOOLEAN FORMULA

- **Input** : A quantified boolean formula:

$$Q_1 x_1 Q_2 x_2 ... Q_n x_n, \varphi(x_1, ..., x_n)$$

$$\text{with } Q_i \in \{\exists, \forall\}, x_i \in \{0, 1\}$$

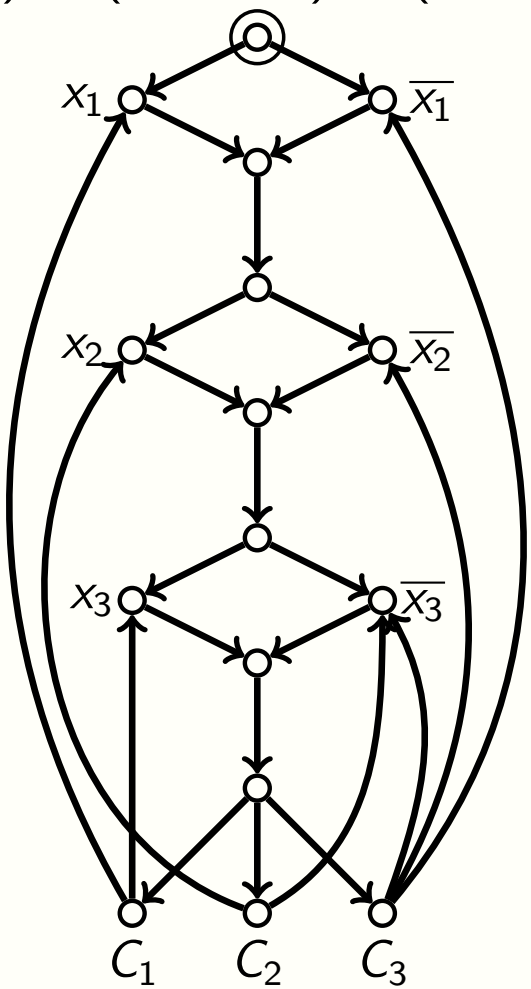
- **Output** : Is the formula true ?

An equivalent problem : QBF-GAME

ED-Geography is PSPACE-complete [Schaeffer 1978]

Reduction from QBF-GAME :

$$(x_1 \vee x_3) \wedge (x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$$



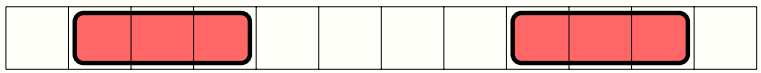
Some nimbers sequences

ArcKayles on a path

0 0 1 1 2 0 3 1 1 0 3 3 2 2 4 0 5 2 2 3 3 0 1 1 3 0 2 1 1 0 4 5 2 7
4 0 1 1 2 0 3 1 1 0 3 3 2 2 4 4 5 5 2 3 3 0 1 1 3 0 2 1 1 0 4 5 3 7...

Period 34 with some finite exceptions up to 52

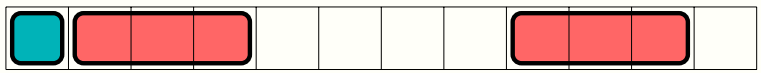
James Bond Game



0 0 0 1 1 1 2 2 0 3 3 1 1 1 0 4...

2^{28} known values, periodicity conjectured

0.106 Game



0 1 0 0 0 1 2 2 2 1 4 4 0 1 0 6...

Period 328226140474, with preperiod 465384263797.

Guy's conjecture: all finite octal games have periodic nimber sequence.

Conclusion

Current research questions ?

- Graphs and Games: combinatorial games version on graphs
- Metatheory: Misère, scoring games, loopy games
- Link with other fields:
  - Artificial Intelligence for generic games
  - Game versions of parameters of graphs
  - Logic, automata theory...

Merci !