Pavages et automates cellulaires
Partie II

École Jeunes Chercheurs
en Informatique Mathématique

N. Aubrun, G. Theyssier

Institut de Mathématiques de Marseille

Lyon, janvier 2016

Forewords

- not a general course on cellular automata

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- recall from part I: space of configuration is compact!

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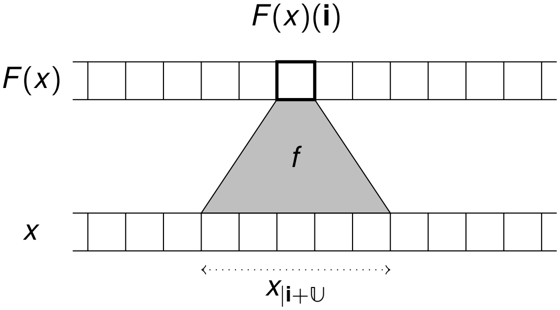
Cellular Automaton of Dimension d

- Q : alphabet
- $\mathbb{U} \subseteq \mathbb{Z}^d$: finite neighborhood
- $f : Q^{\mathbb{U}} \to Q$: local transition map

Cellular Automaton of Dimension d

- Q : alphabet
- $\mathbb{U} \subseteq \mathbb{Z}^d$: finite neighborhood
- $f : Q^{\mathbb{U}} \to Q$: local transition map
- a global map: $F : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$ defined by

$$\forall \mathbf{i} \in \mathbb{Z}^d, F(x)(\mathbf{i}) = f(x_{|\mathbf{i}+\mathbb{U}})$$



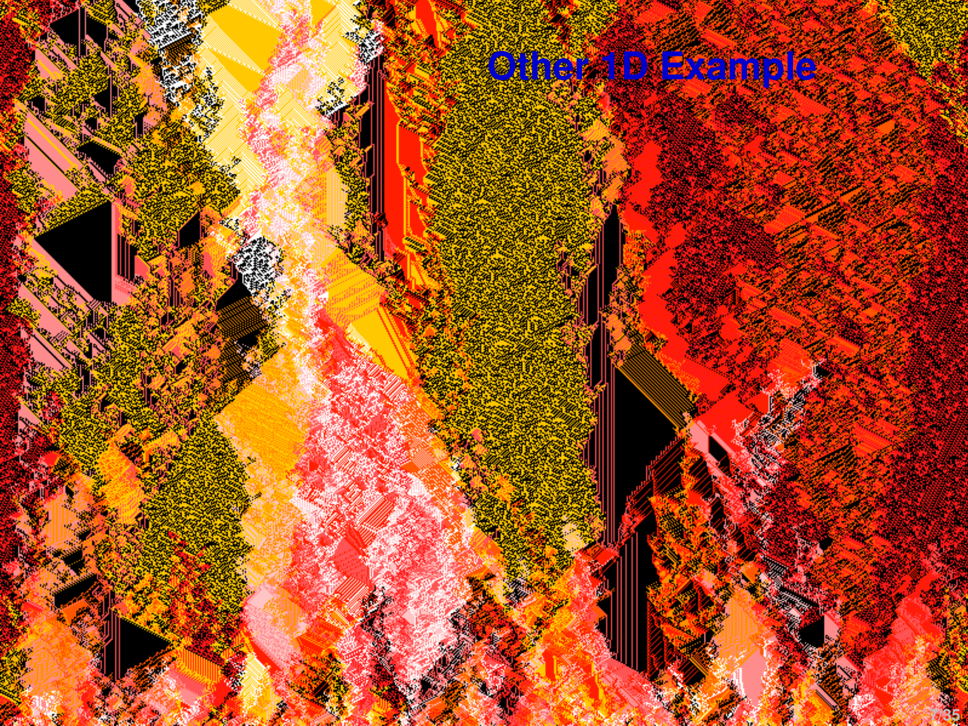
1D Example: XOR

- $d = 1$, $Q = \{0, 1\}$, $\mathbb{U} = \{-1, 0\}$, $f(a, b) = a + b \bmod 2$:

$$F(x)(\mathbf{i}) = f(x_{|\mathbf{i}+\mathbb{U}})$$

$F^8(x)$	0	1	0	0	0	0	0	0	0
$F^7(x)$	0	1	1	1	1	1	1	1	1
$F^6(x)$	0	1	0	1	0	1	0	1	0
$F^5(x)$	0	1	1	0	0	1	1	0	0
$F^4(x)$	0	1	0	0	0	1	0	0	0
$F^3(x)$	0	1	1	1	1	0	0	0	0
$F^2(x)$	0	1	0	1	0	0	0	0	0
$F(x)$	0	1	1	0	0	0	0	0	0
x	0	1	0	0	0	0	0	0	0

Other 1D Example



2D Examples

DEMO...

Hedlund's Theorem

Theorem (Curtis-Lyndon-Hedlund)

Cellular automata are exactly the continuous maps that commute with translations.

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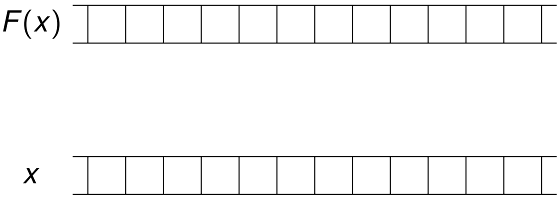

- $\Rightarrow$

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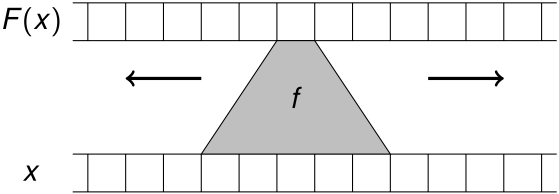

- $\Rightarrow$ translation invariant

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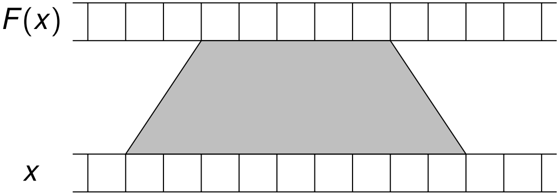

- $\Rightarrow$ translation invariant and continuous

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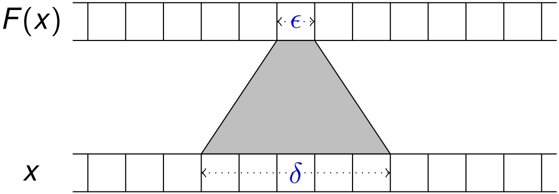

- $\Rightarrow$ translation invariant and continuous
- $\Leftarrow$ uniform continuity: $\forall x \forall \epsilon \exists \delta \Rightarrow \forall \epsilon \exists \delta \forall x$

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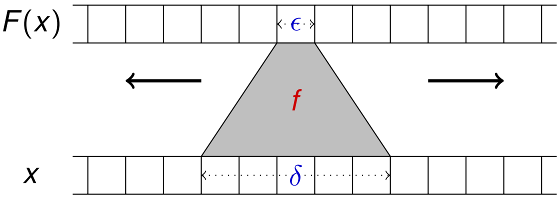

- $\Rightarrow$ translation invariant and continuous
- $\Leftarrow$ uniform continuity: $\forall x \forall \epsilon \exists \delta \Rightarrow \forall \epsilon \exists \delta \forall x$
  + translation invariance

Reversibility

- F global map of a CA
- F can be bijective (one-to-one + onto)

$$F(x) = F(y) \Rightarrow x = y \text{ and } F(Q^{\mathbb{Z}^d}) = Q^{\mathbb{Z}^d}$$

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  1. commutes with shift maps
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Question

What is the neighborhood of F^{-1}?

- F periodic

$$\exists n \forall x : F^n(x) = x$$

Reversibility

Particular cases

- F periodic

$$\exists n \forall x : F^n(x) = x$$

- involution

$$F = F^{-1}$$

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- product of two involutions

$$F = G \circ H \text{ with } G \text{ and } H \text{ involutions}$$

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$$F = G \circ H \text{ with } G \text{ and } H \text{ involutions}$$

$$\Leftrightarrow F = G \circ \underbrace{F^{-1} \circ G}_{H} \text{ with } G \text{ involution}$$

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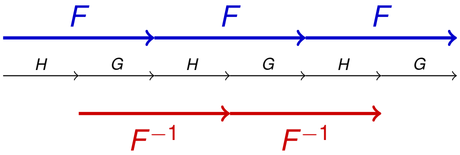
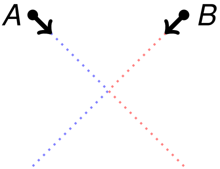


- change arrow of time by an involution

$$I : \overrightarrow{v} \mapsto -\overrightarrow{v}$$



Time Symmetry

Physical intuition

- change arrow of time by an involution

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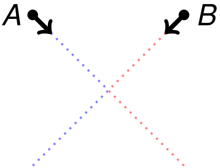

1. run law of physics during time t
2. apply involution I
3. run law of physics during time t
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Time Symmetry

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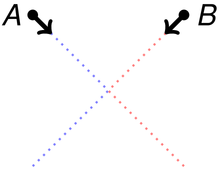

1. run law of physics during time t
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4. apply involution I : **we are back to initial state!**

Time Symmetry

- any permutation π of a finite set E is time symmetric

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Question

Can we decide whether a CA is reversible? periodic? time symmetric?

Periodic configurations

- $\mathbf{i} \in \mathbb{Z}^d$, x is $\mathbf{i}$-periodic if $x(\mathbf{j}) = x(\mathbf{j} + \mathbf{i})$ for all $\mathbf{j}$

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- for 1D CA: F surjective $\Leftrightarrow$ $F_{|P}$ surjective

Open problem

Is there a 2D surjective CA F with $F_{|P}$ not surjective?

Surjunctivity

Theorem

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Proof:

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  1. configurations Q^G
  2. CA $F : Q^G \to Q^G$ with $F(x)(\mathbf{i}) = f(x_{|\mathbf{i}+\mathbb{U}})$

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Open problem (Gottschalk 1973)

$\exists$ group G with some injective CA which is not surjective?

Long Term

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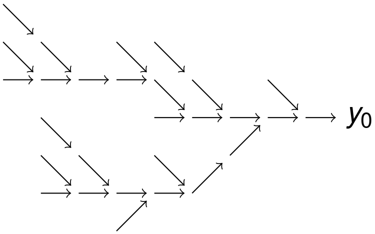

- **nilpotency:** $\exists y_0, \forall x, \exists t : F^t(x) = y_0$

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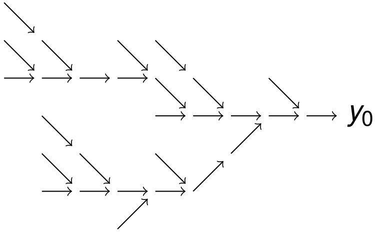

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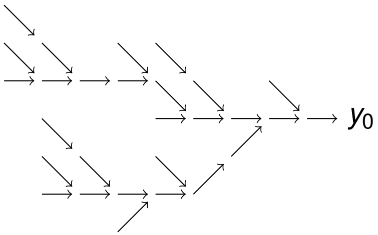

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Question

- can we decide nilpotency? periodic nilpotency?
- are nilpotency and periodic nilpotency equivalent?

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Basic Construction

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If T can tile the plane

- $\exists x \in T^{\mathbb{Z}^2}$ with $F(x) = x$
- $F(\overline{e}) = \overline{e}$
- F is not nilpotent

If T cannot tile the plane

- $\exists N$ s.t. no $N \times N$ pattern is T-valid
- $\forall x \in Q^{\mathbb{Z}^2} : F^N(x) = \overline{e}$
- F is nilpotent

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If T can tile the plane ***periodically***

- $\exists x \in P \cap T^{\mathbb{Z}^2}$ with $F(x) = x$
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- F is not **periodic** nilpotent

If T cannot tile the plane ***periodically***

- $\forall x \in P$ e appears in $F(x)$
- $\forall x \in P, \exists t : F^t(x) = \overline{e}$
- F is **periodic** nilpotent

Consequences

1. there are aperiodic tile sets (Berger, 64)

Theorem

For 2D CA, nilpotency $\not\equiv$ periodic nilpotency

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3. the periodic domino problem is undecidable (Gurevich-Koryakov,72)

Theorem

For 2D CA, periodic nilpotency is undecidable

Determinism

same story with NE-deterministic tile sets...

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3. the deterministic periodic domino problem is undecidable (Mazoyer-Rapaport,98)

Theorem

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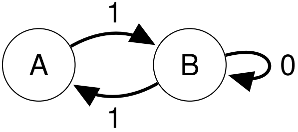
1D Sofic Subshifts

- graph $G = (V, E, \lambda)$ with edge labeling $\lambda : E \to Q$



- Σ_G : labeled bi-infinite paths

$$\cdots 01110110 \cdots \notin \Sigma_G$$
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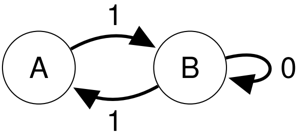


- Σ_G : labeled bi-infinite paths

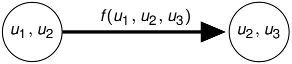
$$\cdots 01110110 \cdots \notin \Sigma_G$$
$$\cdots 011011110 \cdots \in \Sigma_G$$

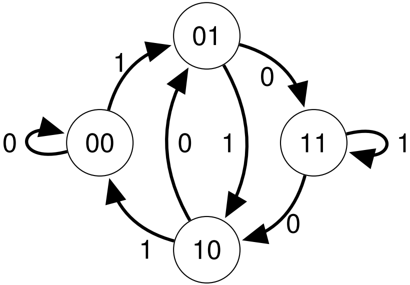
Theorem

- given G it is decidable whether $\Sigma_G = \emptyset$
- given G and G' it is decidable whether $\Sigma_G = \Sigma_{G'}$

1D Sofic Subshifts and CAs

- suppose $\mathbb{U} = \{-1, 0, 1\}$
- The subshift $F(Q^{\mathbb{Z}})$ is sofic

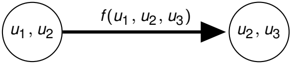


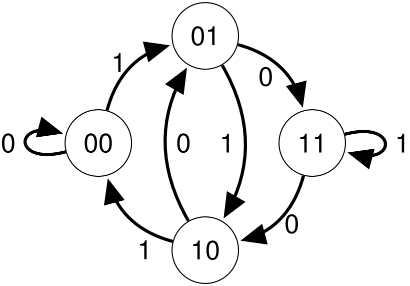


$f(a, b, c) = a + b + c \bmod 2$

1D Sofic Subshifts and CAs

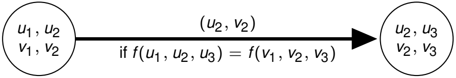
- suppose $\mathbb{U} = \{-1, 0, 1\}$
- The subshift $F(Q^{\mathbb{Z}})$ is sofic





$f(a, b, c) = a + b + c \bmod 2$

- $\Sigma_R = \{(x, y) : F(x) = F(y)\}$ (abusing notation) is sofic



1D CA and first-order theory

- **surjectivity:** $\forall y, \exists x : F(x) = y$
- **injectivity:** $\forall x, x' : F(x) = F(x') \Rightarrow x = x'$

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Theorem (Amoroso-Patt,72)

Injectivity(=reversibility) and surjectivity are decidable in 1D.

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$$F(Q^{\mathbb{Z}}) = Q^{\mathbb{Z}}$$
$$\{(x, y) : F(x) = F(y)\} = \{(x, x) : x \in Q^{\mathbb{Z}}\}$$

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- any first-order property is decidable (Sutner,07)

Table of contents

Directed Tiles and Snakes

- T a tile set

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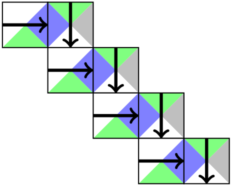
- orientation on tiles $\vec{D} : T \to \{(\pm 1, 0), (0, \pm 1)\}$

Directed Tiles and Snakes

- T a tile set

- orientation on tiles $\vec{D} : T \to \{(\pm 1, 0), (0, \pm 1)\}$

- **directed snake**: path following arrows and respecting colors



- $s_n \in T$
- $p_{n+1} = p_n + \vec{D}(s_n)$

Infinite Directed Snake Problem

Infinite Directed Snake Problem

- **decision problem:** given a directed tile set $(T, \vec{D})$, does it admit an infinite directed snake?

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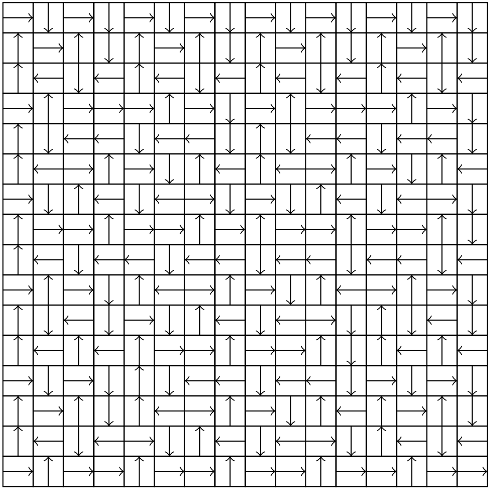
Theorem (Kari, 1994)

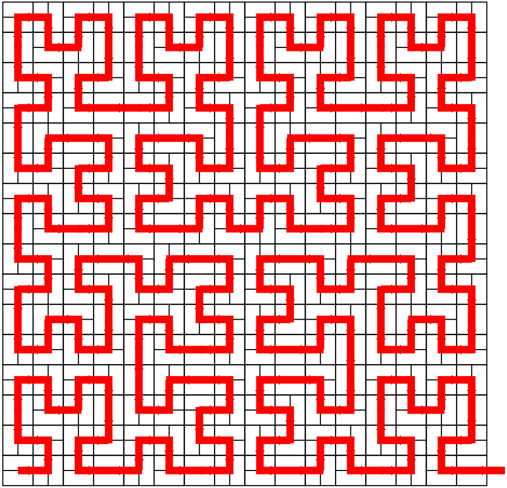
The infinite directed snake problem is undecidable

A Space Filling Snake Tileset



- $\exists$ directed tile set such that **all snakes look like this**

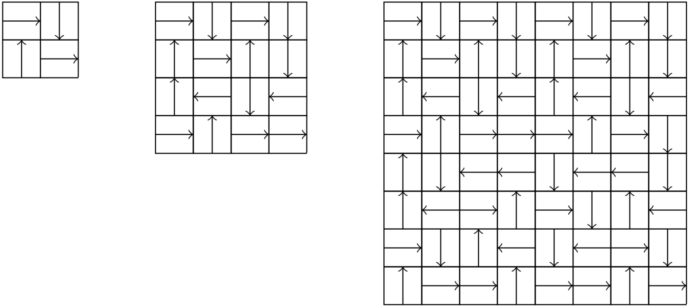
A Space Filling Snake Tileset



- ∃ directed tile set such that **all snakes look like this**

A Space Filling Snake Tileset

- **idea 1:** Hilbert curve as a substitution σ



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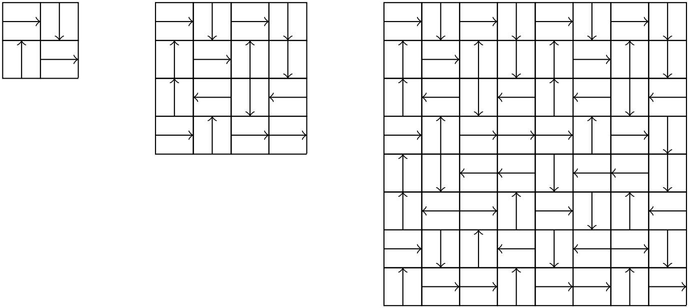

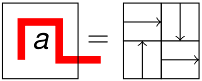

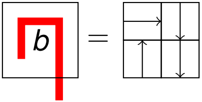

- $\times 2$ mirror
- $\times 4$ rotate
- $|Q| = 16$

$$\sigma(q) = \begin{array}{|c|c|} \hline NW_q & NE_q \\ \hline SW_q & SE_q \\ \hline \end{array}$$

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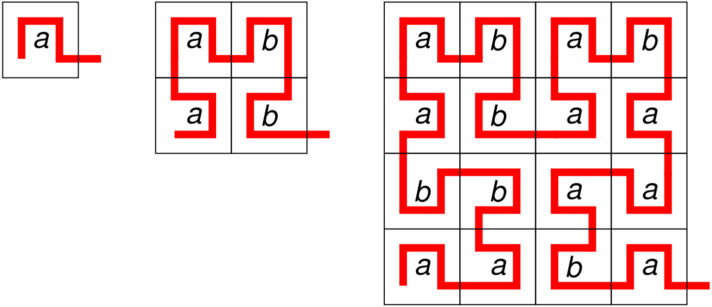

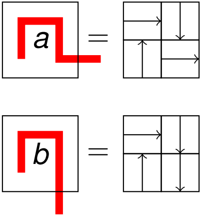

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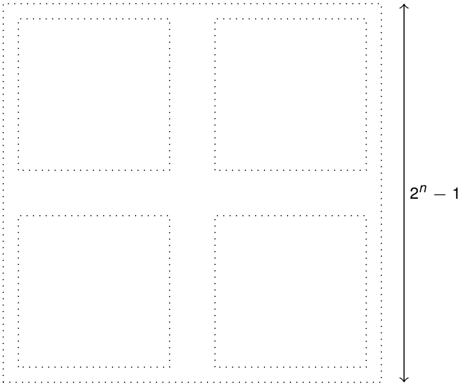
A Space Filling Snake Tileset

- **idea 2:** self-similar structure to execute σ



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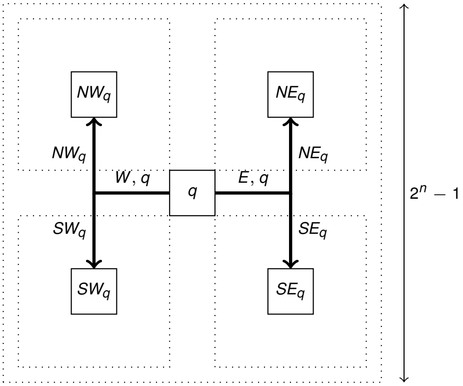

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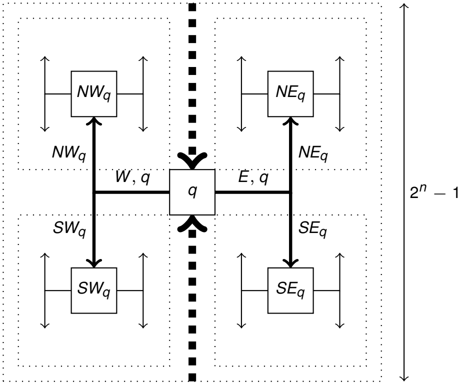

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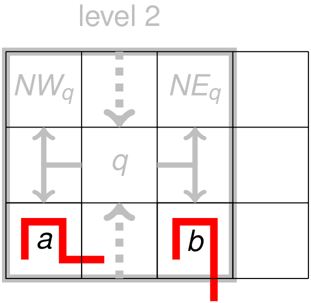
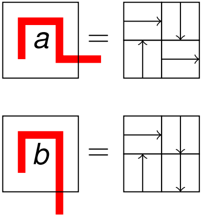
A Space Filling Snake Tileset

- final touch at squares of level 1 and 2





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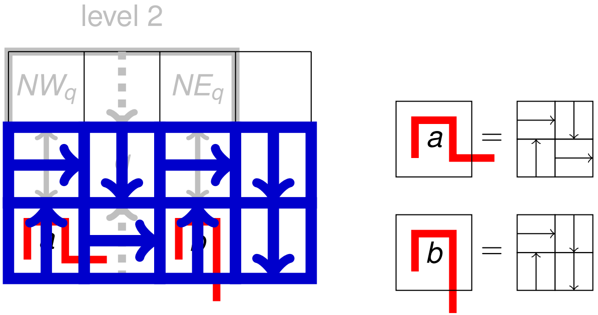

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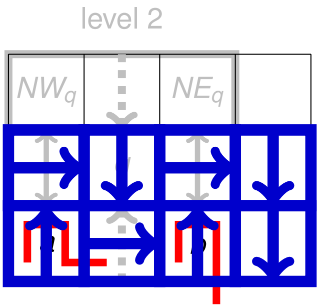

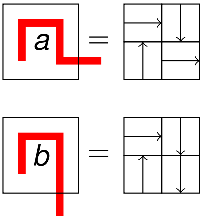

- **idea 3:** snake $\Rightarrow$ correct tiling

Infinite Snake Problem Undecidable

Proof:

Infinite Snake Problem Undecidable

Proof:

1. take a space filling directed tile set $(T, \vec{D})$
   - T tiles the plane and $(T, \vec{D})$ admits infinite snakes
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3. if T' tiles the plane then T'' admits an infinite snake
4. if T' does not tile the plane
   - suppose s is an infinite snake of T''
   - induces an infinite snake of T
   - must cover arbitrarily large $N \times N$ squares
   - so arbitrary large $N \times N$ squares are tiled by T'
   - by compacity T' admits a tiling of the plane: **contradiction!**

Consequences

- given a directed tile set $T, \vec{D}$, define F on $T \times \{0,1\}$:
  1. F computes 1D XOR CA along snakes
  2. F does nothing when tiling error in the neighborhood

$$F(x)(\mathbf{i}) = \begin{cases} x_{\mathbf{i}} & \text{if } T\text{-layer invalid at } \mathbf{i}, \\ (\tau_{\mathbf{i}}, b) & \text{else with } b = b_{\mathbf{i}} + b_{\vec{D}(x_{\mathbf{i}})} \bmod 2, \end{cases}$$

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- global bound on snakes
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Undecidability!

Theorem (Kari 94 + Gajardo-Kari-Moreira 12)

For 2D CA each of reversibility/periodicity/time-symmetry is undecidable

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Open

Is time symmetry decidable in 1D?

This is Not the End

- from 2D SFT to 1D CA: is it about determinism?
- deterministic chaos / topological dynamics
- ergodic dynamics / stochastic CA
- **intrinsic simulations and universality**
- **CA as a parallel computational model**
- links between **blue** and red