Logic, Automata and Games

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Introduction [1]

- Games provide a powerful framework for understanding interactions.
- They are present in various features of Computer Science: e.g. *alternating machines, reactive systems, games semantics* [1].
- Here we are only interested in a very peculiar use of games: the purpose is to

  elucidate the topological complexity of languages of infinite words recognized by automata.

Introduction [2]

Definition

Given any (finite) non-empty set A,

- A^* denotes the set of all *finite* words on A
- ε denotes the empty word
- A^ω denotes the set of all *infinite* words on A
- the concatenation of two finite words u and v is denoted by uv
- we use
  - a, b for the letters of the alphabet,
  - u, v for the finite words,
  - $\vec{a}, \vec{b}$ for the infinite words.

Introduction [3]

Definition

A Büchi automaton [10] is of the form

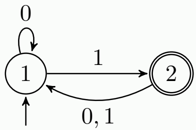
$$\mathscr{A} = (A, Q, q_i, \Delta, F)$$

where

1. A is a finite alphabet
2. Q is a finite state of states
3. q_i is the initial state
4. $\Delta \subseteq Q \times A \times Q$,
5. $F \subseteq Q$ stands for the set of accepting states.

Example

A Büchi automaton



- It is deterministic when

$$\Delta \text{ is a function } Q \times A \to Q.$$

i.e. for all $(q,a) \in Q \times A$

$$\text{there exists a unique } q' \in Q \text{ such that } (q,a,q') \in \Delta.$$

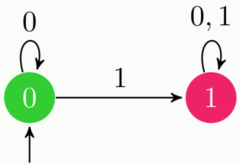
- $\mathscr{A}$ accepts an infinite word $\vec{a} = a_0 a_1 a_2 \dots$ if there exists some run $\rho_{\vec{a}} \in Q^\omega$ that visits infinitely often some accepting state.

  $\rho_{\vec{a}}$ must verify
  - $\rho_{\vec{a}}(0) = q_i$ and for each integer n,
  - $\big(\rho_{\vec{a}}(n), a_n, \rho_{\vec{a}}(n+1)\big) \in \Delta$.

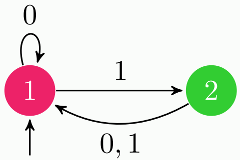
- Parity automata are defined similarly except for the acceptance condition which replaces F with a mapping $c : Q \to \mathbb{N}$.
  Then, an infinite word $\vec{a}$ is accepted if there exists existe a run $\rho_{\vec{a}}$ s.t.
  $$\limsup_{n\to\infty} c\big(\rho_{\vec{a}}(n)\big) \text{ is even [10, 4].}$$
  i.e. $\vec{a}$ is accepted iff there exists some run s.t. the set S of the states that are visited infinitely often satisfies
  $$\max\{c(q) \mid q \in S\} \text{ is even.}$$

Example

- **?**



- **?**



- The language recognized by an automaton is the set of words it accepts.
- Parity automata, Büchi automata and deterministic parity automata recognize the same class of languages:

ω-*regular languages*.

- If $\mathscr{A} = (A, Q, q_i, \delta, c)$ is some deterministic parity automaton, then

$$\mathscr{A}^{\complement} = \left(A, Q, q_i, \delta, c'\right)$$

where c' is defined by $c'(n) = c(n) + 1$ satisfies

$$\mathscr{L}(\mathscr{A})^{\complement} = \mathscr{L}(\mathscr{A}^{\complement}).$$

- We will make use of the set theoretical definition of a tree: a tree T on an alphabet A is a set $T \subseteq A^*$ closed under prefixes.

Finite two-player games with perfect information [1]

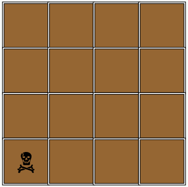
We only take into account games s.t.:

- there are *two players*;
- plays are both *sequential* (no simultaneous moves, players take turns) and finite;
- information is *perfect* (at any time, the whole configuration of the play is accessible to all players, i.e nothing is hidden, no chance).
- when a play is over, there is a *winner* and a *looser*.
  - $\neq$Poker
  - $\neq$Battleship
  - $\neq$Game of the goose
  - $\neq$Chess
  - $\neq$Checkers

Finite two-player games with perfect information [2]

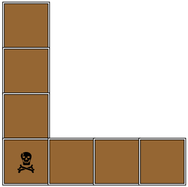
Example (The chocolate bar)

- Two players ($\mathbf{0}$ and $\mathbf{1}$) take turns munching chocolate.
- Player $\mathbf{0}$ starts.
- Each time players must eat a piece of chocolate. But when then take out a piece (i, j),they must also take out all (i', j') such that $i' \geq i$ and $j' \geq j$.
- Unfortunately, the bottom piece $(0, 0)$ is lethal. The one who dies, loses the game, the other one wins.



Finite two-player games with perfect information [3]

- If player $\mathbf{0}$ does the following:
  1. at first move, $\mathbf{0}$ eats out $(1,1)$, so that the opponent is left with



  2. then, each time $\mathbf{0}$ must play, if the opponent takes out piece $(0,i)$, resp. $(i,0)$, player $\mathbf{0}$ picks the symmetrical piece $(i,0)$, resp. $(0,i)$.
- This makes sure that player $\mathbf{1}$ munches the bottom piece $(0,0)$ and dies.

We just showed that player $\mathbf{0}$ – the one who starts to play – has a way of playing – that only depends on its adversary's move– that guarantees its victory[1] i.e. we exhibited a

winning strategy for player $\mathbf{0}$.

- All possible moves of such a game form a well-founded labeled tree
  - Each node corresponds to some configuration – the root being the initial configuration.
  - Each branch – from the root to some leaf – represents a possible play.

Finite two-player games with perfect information [5]

Example (Chess)

- The initial configuration is the chessboard with the initial positions of the various pieces and the fact that White must play.
- The immediate successors of the initial configuration are all the configurations that White may reach in one move. (8 pawns + 2 knights; 2 moves each).

Finite two-player games with perfect information [6]

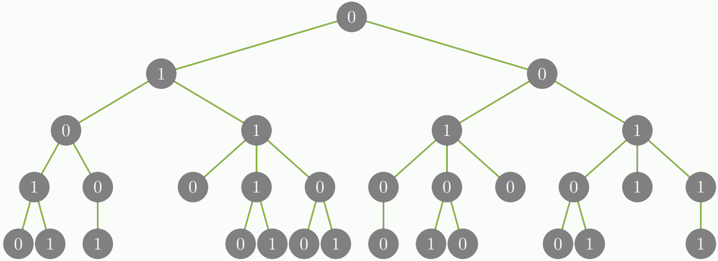
Definition (Finite Game Tree)

A *finite game tree* (T, e) is a well-founded non-empty tree T, labeled by some mapping $e : T \to \{\mathbf{0}, \mathbf{1}\}$. The two-player game with perfect information associated with (T, e) consists in

- placing a token on the root of the tree, and
- for each node on which the token stands, player $e(n)$ loses the game if n is leaf, otherwise pushes the token to any immediate successor of node of n.

Finite two-player games with perfect information [7]

Example (Game tree)



Finite two-player games with perfect information [8]

Definition

A *strategy* for player $\mathbf{0}$ in the game associated with a finite tree (T, e) is a non-empty labeled tree (σ, e) satisfying

- $\sigma \subseteq T$,
- each leaf of σ is also a leaf of T,
- for each node $n \in \sigma$ that is not a leaf:
  - if $e(n) = 0$, then a unique immediate successor of of n belongs to σ;
  - if $e(n) = 1$, then every immediate successor of n belong to σ.

A strategy for $\mathbf{1}$ is defined *mutatis mutandis*.

Finite two-player games with perfect information [9]

- We say a player applies a strategy if the game is restricted to this strategy.
- If player $\mathbf{0}$ applies a strategy σ and player $\mathbf{1}$ applies a strategy τ, then the game restricts itself to a unique play: the tree whose only branch is

$$\sigma \cap \tau.$$

Example (Strategy for $\mathbf{0}$)

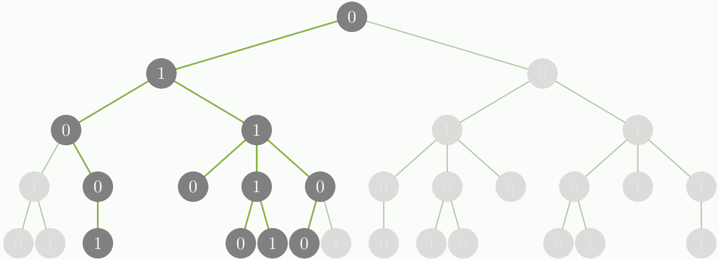


Figure: *A strategy for player* $\mathbf{0}$ *in the previous game.*

Example (Strategy for $\mathbf{1}$)

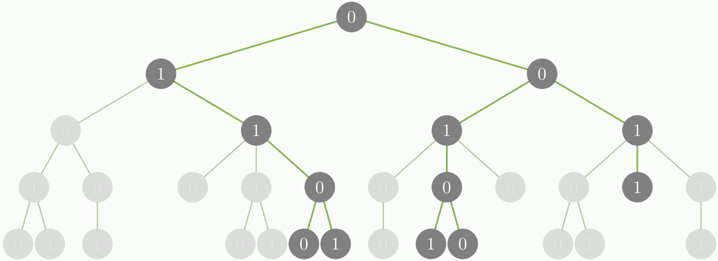


Figure: *A strategy for player* $\mathbf{1}$ *in the previous game.*

Finite two-player games with perfect information [12]

Definition

In the game associated with a finite tree (T, e),

- a *strategy* σ for player $\mathbf{0}$ is *winning* if for every leaf $f \in \sigma$, $e(f) = 1$;
- a *strategy* τ for player $\mathbf{1}$ is *winning* if for every leaf $f \in \sigma$, $e(f) = 0$;

Definition

A game is *determined* if one of the players has a winning strategy.

Finite two-player games with perfect information [13]

- For certain class of games, the fact that games are determined is a very strong statement. It transforms a negative assertion in a positive one:
    - (player $\mathbf{J}$ has no w.s.) $\Longrightarrow$ (player $\mathbf{1}-\mathbf{J}$ has a w.s.).
- Henceforth, a determinacy principle is a highly non constructive statement.
    - it is claimed that a w.s. exists for a given player without being able to construct even one.

Finite two-player games with perfect information [14]

Example (Rectangular Chocolate Bar)

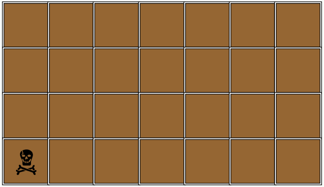


Figure: (n, m) *chocolate bar.*

- We know that if $n = m \neq 0$, then player $\mathbf{0}$ has a w.s.
- Assuming that this game is *determined* – confirmed by Theorem **??** – we show that player $\mathbf{0}$ has a w.s. whatever the size $(n, m) \neq (0, 0)$.

Finite two-player games with perfect information [15]

- Since this game is determined, in order to show that $\mathbf{0}$ has a w.s. it is enough to show that its opponent does not have one.
- We proceed by contradiction and assume that player $\mathbf{1}$ has a w.s. τ and we build a w.s. σ for player $\mathbf{0}$.

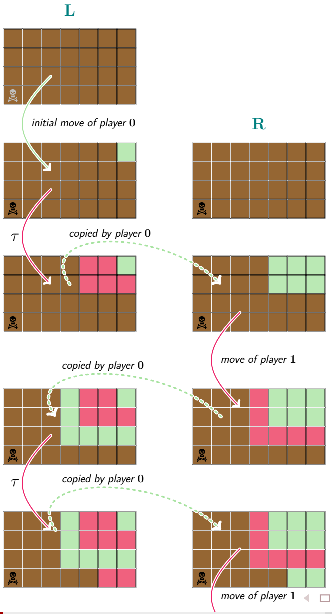
We consider two different plays : L and R.

- In L, player $\mathbf{1}$ applies a w.s. τ.
- In R, player $\mathbf{1}$ plays freely. Player $\mathbf{0}$ applies a strategy σ.

We define σ by:

- L_0 : player $\mathbf{0}$ eats up (n, m);
- L_{2i+1} is the answer by τ to L_{2i} of player $\mathbf{0}$;
- R_{2i} is a copy by player $\mathbf{0}$ of player $\mathbf{1}$'s L_{2i+1}
- R_{2i+1} is any free choice by $\mathbf{1}$.

Finite two-player games with perfect information [16]



Evaluation Game [1]

- Although they may seem abstract, these games lie at the core of what it takes to evaluate a formula:
  - to check whether or not holds true in a given model, comes down to solving a game.
- One particular example is evaluation games for 1st order logic [2].

Evaluation Game [2]

Definition

Let $\mathscr{L}$ be a 1st order language,

- $\mathscr{M}$ an $\mathscr{L}$-structure and
- ϕ a closed $\mathscr{L}$-formula whose connectors are among $\{\neg, \vee, \wedge\}$.

We define the *evaluation game* $\mathbb{EV}(\phi, \mathscr{M})$ as a finite two-player game with perfect information.

- Players are called
  - **V**erifier
  - **F**alsifier.
- Moves are defined by:

Evaluation Game [3]

Definition

if ϕ is	who's turn	goes on with
atomic	no one	play stops
$\exists x\, \psi$	**V** picks a in the domain of $\mathscr{M}$	$\psi_{[a/x]}$
$\forall x\, \psi$	**F** picks a in the domain of $\mathscr{M}$	$\psi_{[a/x]}$
$(\phi_0 \vee \phi_1)$	**V** choses $i \in \{0,1\}$	ϕ_i
$(\phi_0 \wedge \phi_1)$	**F** choses $i \in \{0,1\}$	ϕ_i
$\neg\psi$	**V** and **F** switch roles	ψ

By construction, one stops on an atomic formula of the form $R(t_1,\ldots,t_n)_{[a_1/x_1,\ldots,a_k/x_k]}$ where $x_1,\ldots,x_k$ are all variables from $R(t_1,\ldots,t_n)$ and $a_1,\ldots,a_k$ are elements from $|\mathscr{M}|$.

Evaluation Game [4]

Definition

Verifier wins iff

$$\left(t_1{}^{\mathscr{M}}_{[a_1/x_1,\ldots,a_k/x_k]},\ldots,t_n{}^{\mathscr{M}}_{[a_1/x_1,\ldots,a_k/x_k]}\right)\in R^{\mathscr{M}}.$$

- The rules are defined in order to obtain:

Theorem

If $\mathscr{L}$ is a 1st order language, $\mathscr{M}$ any model, ϕ any $\mathscr{L}$-formula whose connectors are among $\{\neg,\vee,\wedge\}$. Then

$$\textit{Verifier has a w.s. in } \mathbb{EV}(\phi,\mathscr{M}) \iff \phi \textit{ holds true in } \mathscr{M}.$$

Infinite two-player games with perfect information [1]

- Going from finite to infinite games is a giant leap.
- Everything becomes less easy and more technical since topological notions are required.
- Among all the infinite two-player games with perfect information, one stands out: the Gale-Stewart game.

Definition

Given $L \subseteq A^\omega$, the Gale-Stewart game $\mathscr{G}(L)$ is an infinite game in which the players (*I* and *II*) alternately chose $a \in A$. Player *I* starts. Player *I* wins iff the infinite word $\vec{a}$ constructed during the play satisfies $\vec{a} \in L$. Otherwise, player *II* wins.

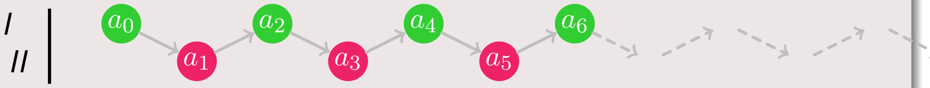


Figure: *Gale-Stewart Game.*

Firstly consider for each non-null integer n, a finite version $\mathscr{G}_n(M)$ for $M \subseteq A^{2n}$. Clearly, these games are determined. Not only because these

Infinite two-player games with perfect information [3]

are finite two-player games with perfect information, but also because the formula that expresses that *I* does not have a w.s.:

$$\neg\exists a_0 \forall a_1 \exists a_2 \forall a_3 \ldots \forall a_{2n-1} \quad \vec{a} \in M$$

is *logically* equivalent to the formula

$$\forall a_0 \exists a_1 \forall a_2 \exists a_3 \ldots \exists a_{2n-1} \quad \vec{a} \notin M$$

which says that *II* has a w.s..
Gale-Stewart determinacy can be regarded as a generalisation of this phenomenon to the "*infinite formula*" describing the existence of a w.s. for player *I*. Indeed determinacy claims that if *I* does not have a w.s., i.e.

$$\neg\exists a_0 \forall a_1 \exists a_2 \forall a_3 \ldots\ldots \quad \vec{a} \in L,$$

then player *II* has one:

$$\forall a_0 \exists a_1 \forall a_2 \exists a_3 \ldots\ldots \quad \vec{a} \notin L.$$

Infinite two-player games with perfect information [4]

- However, contrary to what happens in the finite case, determinacy is not a simple statement in the infinite one.
  - One can show there exist non-determined games (this requires the *Axiom of Choice*.)
  - On can show these games are determined for a large class of sets (the *Borel* sets).

A non-determined game [1]

Definition (Banach-Mazur Game)

Given $L \subseteq A^\omega$, the Banach-Mazur game $\mathscr{B}(L)$ is identical to the Gale-Stewart game $\mathscr{G}(L)$ except that players play *non-empty* words $(\in A^*)$ instead of letters $(\in A)$.
Player *I* wins if the concatenation $\vec{a}$ of the words played satisfies $\vec{a} \in L$.
Otherwise *II* wins.

Figure: *Banach-Mazur Game.*

A non-determined game [2]

- Given any set A and $L \subseteq A^\omega$, one can easily define A' and $L' \subseteq A'^\omega$ such that the game $\mathscr{G}(L')$ simulate the game $\mathscr{B}(L)$ so that the existence of a w.s. for a player in the first game induces the existence of a w.s. for the same player in the second game.
- Therefore, to show that there exists L' s.t. $\mathscr{G}(L')$ is not determined, it is enough to come up with a set L such that $\mathscr{B}(L)$ is not determined.

Definition

$F \subseteq \{0,1\}^\omega$ is a *flip set* if for all $\vec{x}, \vec{y} \in \{0,1\}^\omega$, si $\exists k \ \big(x_k \neq y_k \wedge \forall n \neq k \ (x_n = y_n)\big)$, i.e. $\vec{x}$ and $\vec{y}$ only differ by a *single* digit , then $x \in F \iff y \notin F$.

A non-determined game [3]

Proposition (AC)

If $F \subseteq \{0,1\}^\omega$ is a flip set, then the game $\mathscr{B}(F)$ is not determined.

Towards a contradiction, we assume that player *II* has a w.s. τ which he applies in the lower play, and we show that player *I* also has a w.s. in the upper play.

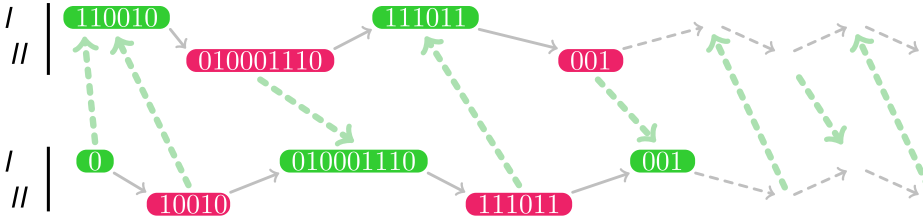


Figure: *Player II applies strategy τ in the lower play*

A non-determined game [4]

Similarly one shows that if Player I has a w.s., then II also has one.

Borel Sets [1]

- there exists a vast class of sets for which all Gale-Stewart games are determined. Its definition is topological: the class of *Borel sets*
  - We equip the set A^ω with the usual topology (the product topology of the discrete topology on A):
    - *Basic open sets* are of the form $N_u = uA^\omega$ for $u \in A^+$.
    - *Open sets* are then of the form $\bigcup_{u \in U} N_u$ for any set $U \subseteq A^*$
- ($U = \emptyset$ and $U = A$ respectively yield $\emptyset$ and A^ω.)
- $\mathbb{N}^\omega$ is similar to $\mathbb{R}$ (equipped with the usual topology: basic open sets are of the form $]x, y[$) since it is homeomorphic – i.e. isomorphic with regard to the topological structure – to $\mathbb{R} \setminus \mathbb{Q}$.

$$\mathbb{N}^\omega \cong \mathbb{R} \setminus \mathbb{Q}$$

Borel Sets [2]

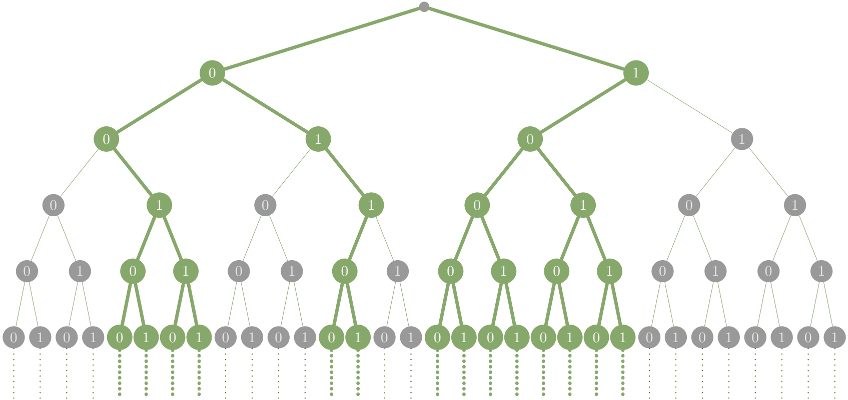


Figure: *An open subset of* $\{0,1\}^\omega$.

Borel Sets [3]

Definition

The class of *Borel* subsets of A^ω is the least that

- contains the *open* sets, and
- is *closed under*
  1. **countable** *union* and
  2. *complementation*.

.

Borel Sets [4]

Example

Borel subsets of $\{0,1\}^\omega$:

1. $\{0^\omega\}$: it is a closed set (the complement of an open set) since

$$\{0^\omega\}^\complement = \bigcup_{u \in \{0,1\}^* 1} N_u$$

2. $\{0,1\}^* 1^\omega$: since it is

$$\bigcup_{u \in \{0,1\}^*} u1^\omega$$

(a countable union of closed sets).

- Given any tree $T \subseteq A^*$, $[T]$ denotes the set of its infinite branches

$$[T] = \{\vec{a} \in A^\omega \mid \forall n \in \mathbb{N}\ \vec{a} \restriction n \in T\}.$$

Borel Sets [5]

- Notice that $[T]^{\complement} = \underbrace{\bigcup_{v \notin T} vA^{\omega}}_{open}$.
- For any $B \subseteq A^{\omega}$

$$B \text{ is } \textit{closed} \iff B = [T]$$

  for some tree $T \subseteq A^*$ [6].

- As soon as they were introduced, the Borel sets were set up in a nice hierarchy by Baire.

Borel Sets [6]

Borel Sets [7]

Definition (Borel Hierarchy)

By induction on ordinals, we define

- $\mathbf{\Sigma}^0_1 = \{\mathit{open}\}$
- $\mathbf{\Pi}^0_\alpha = \{E^\complement \mid E \in \mathbf{\Sigma}^0_\alpha\}$
- $\mathbf{\Sigma}^0_\alpha = \left\{ \bigcup_{n\in\mathbb{N}} E_n \mid E_n \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta \right\}$
- $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$.

$$\mathscr{B} = \bigcup_{\alpha\in On} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha\in On} \mathbf{\Pi}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$$

Borel Sets [8]

Example

Ainsi, parmi les sous-ensembles de l'espace $\{0,1\}^\omega$,

- $\{0^\omega\} \in \mathbf{\Pi}^0_1$
- $\{0,1\}^* 1^\omega \in \mathbf{\Sigma}^0_2$
- $\big(\{0,1\}^* 0\big)^\omega \in \mathbf{\Pi}^0_2$

Indeed,

$$\{0,1\}^\omega = [\{0,1\}^*]$$

$$\{0,1\}^* 1^{\omega\complement} = \big(\{0,1\}^* 0\big)^\omega$$

and

$$\big(\{0,1\}^* 0\big)^\omega = \underbrace{\bigcap_{n\in\mathbb{N}} \underbrace{\big(\{0,1\}^* 0\big)^n \{0,1\}^\omega}_{\mathbf{\Sigma}^0_1}}_{\mathbf{\Pi}^0_2}.$$

$$
\begin{tikzcd}[column sep=small]
 & \vdots & \\
\mathbf{\Sigma}^0_4 & & \mathbf{\Pi}^0_4 \\
 & \mathbf{\Delta}^0_4 \arrow[ul] \arrow[ur] & \\
\mathbf{\Sigma}^0_3 \arrow[ur] & & \mathbf{\Pi}^0_3 \arrow[ul] \\
 & \mathbf{\Delta}^0_3 \arrow[ul] \arrow[ur] & \\
\mathbf{\Sigma}^0_2 \arrow[ur] & & \mathbf{\Pi}^0_2 \arrow[ul] \\
 & \mathbf{\Delta}^0_2 \arrow[ul] \arrow[ur] & \\
\mathbf{\Sigma}^0_1 \arrow[ur] & & \mathbf{\Pi}^0_1 \arrow[ul]
\end{tikzcd}
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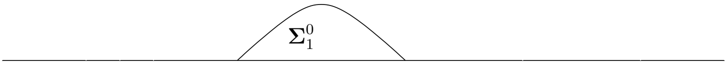
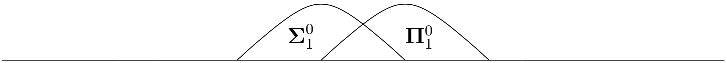
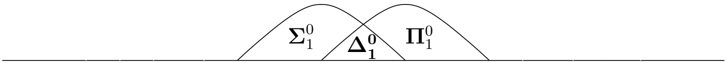
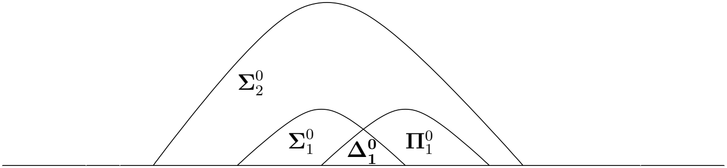

$\mathbf{\Sigma}^0_1$


$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Pi}^0_1$


$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Delta}^0_1$
$\boldsymbol{\Pi}^0_1$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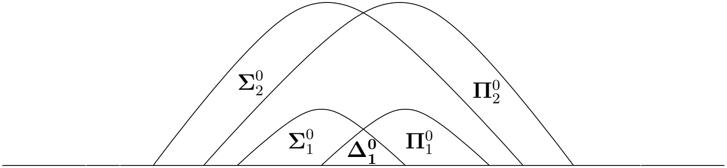
Σ^0_1
$\mathbf{\Delta^0_1}$
Π^0_1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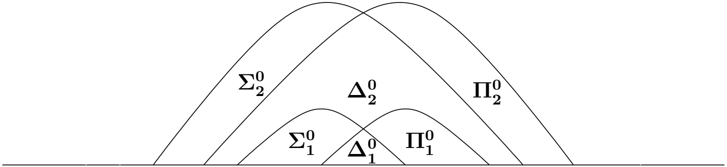
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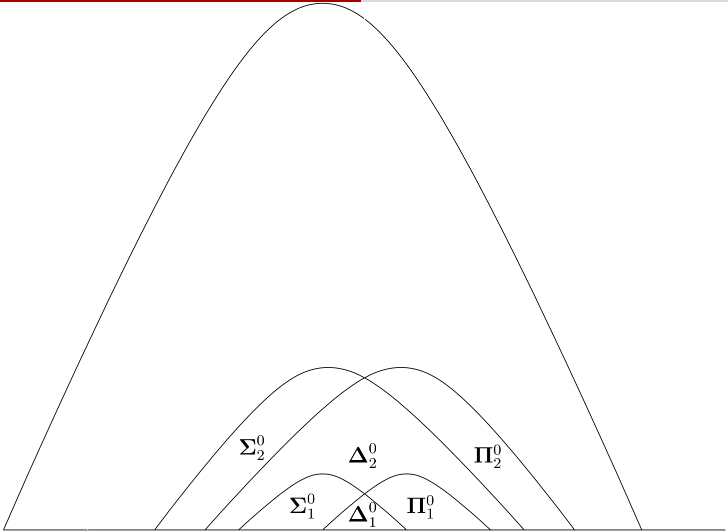
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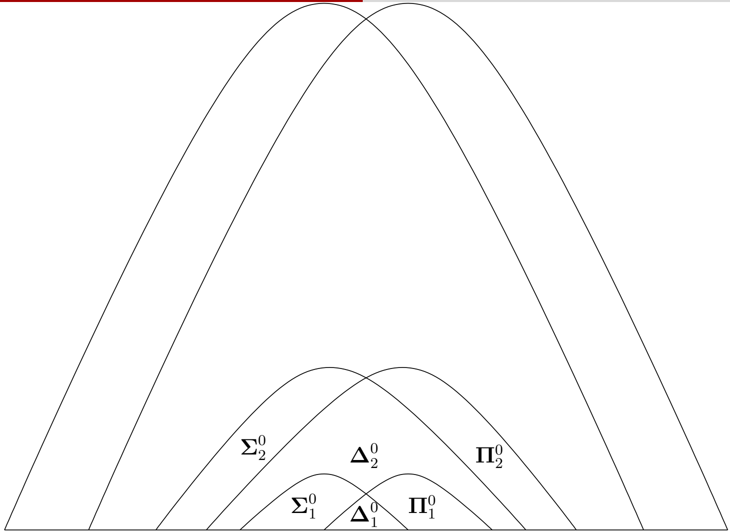
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$\mathbf{\Delta}^0_1$
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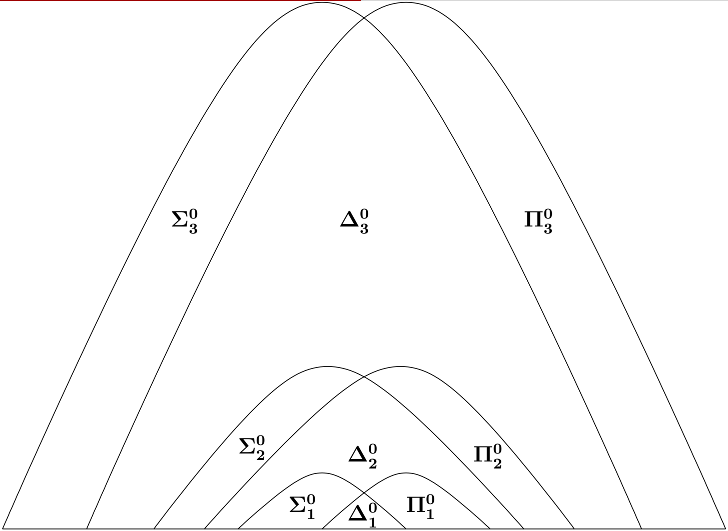

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$\mathbf{\Delta}^0_3$
$\mathbf{\Pi}^0_3$
$\mathbf{\Sigma}^0_2$
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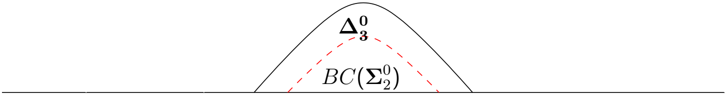
$\mathbf{\Delta}^0_1$
$\mathbf{\Pi}^0_1$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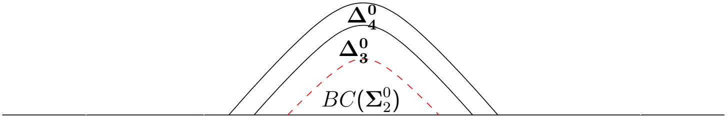
$BC(\mathbf{\Sigma}^0_2)$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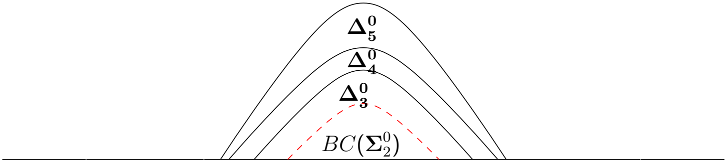
$\mathbf{\Delta}^0_3$
$BC(\mathbf{\Sigma}^0_2)$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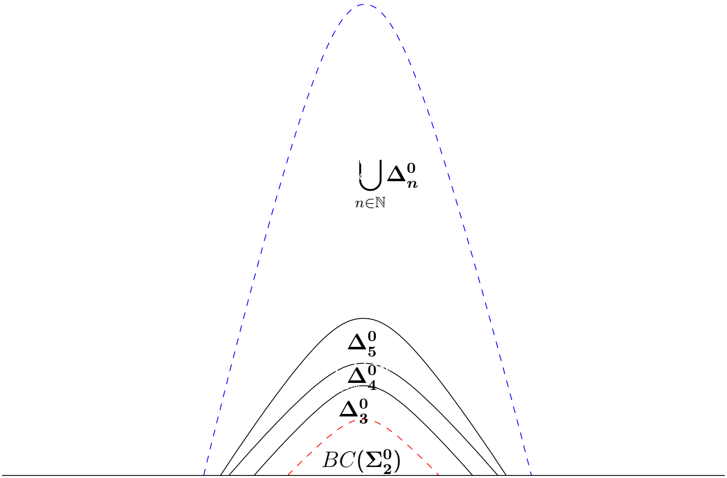
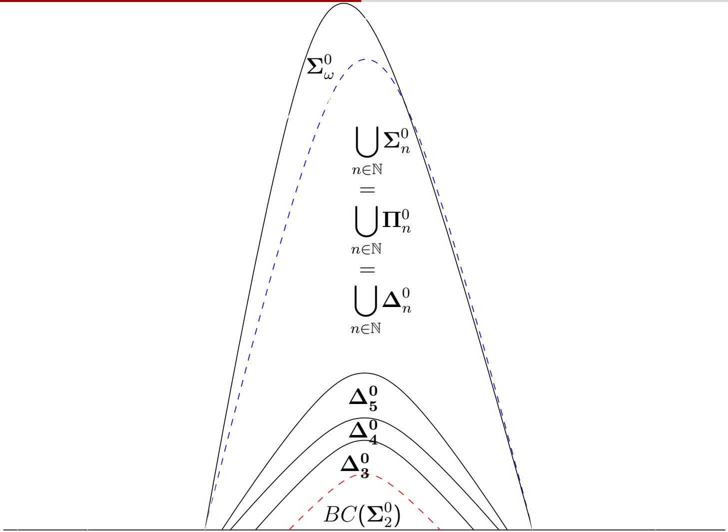
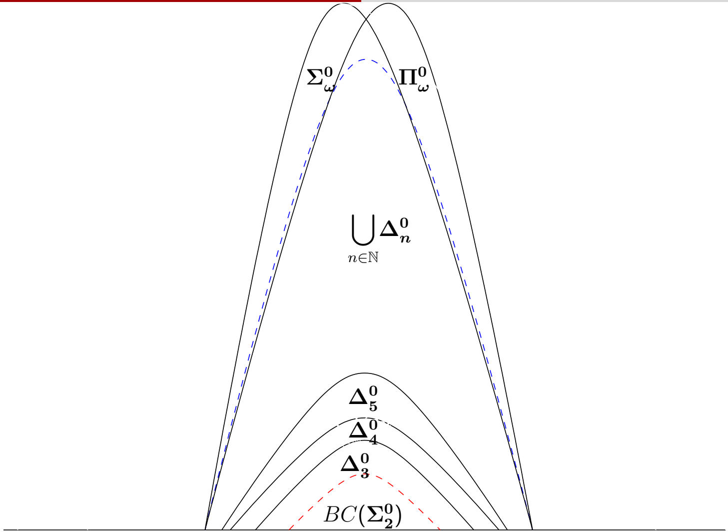
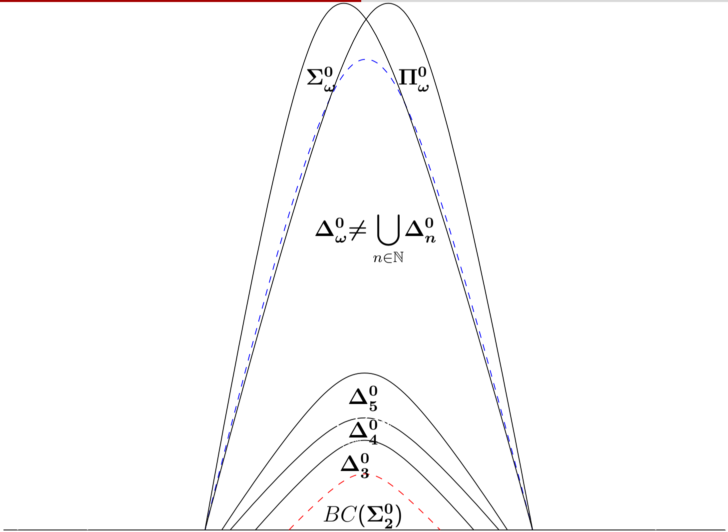
$\mathbf{\Delta^0_4}$
$\mathbf{\Delta^0_3}$
$BC(\mathbf{\Sigma}^0_2)$

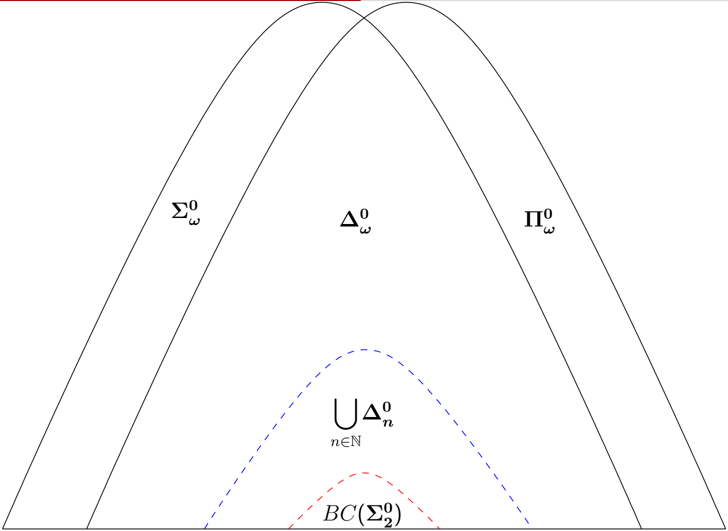

$\bigcup_{n\in\mathbb{N}} \boldsymbol{\Delta}^0_n$
$\boldsymbol{\Delta}^0_5$
$\boldsymbol{\Delta}^0_4$
$\boldsymbol{\Delta}^0_3$
$BC(\boldsymbol{\Sigma}^0_2)$

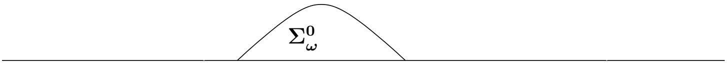

$\mathbf{\Sigma}^0_\omega$
$\bigcup_{n\in\mathbb{N}} \mathbf{\Sigma}^0_n = \bigcup_{n\in\mathbb{N}} \mathbf{\Pi}^0_n = \bigcup_{n\in\mathbb{N}} \mathbf{\Delta}^0_n$
$\mathbf{\Delta^0_5}$
$\mathbf{\Delta^0_4}$
$\mathbf{\Delta^0_3}$
$BC(\mathbf{\Sigma}^0_2)$

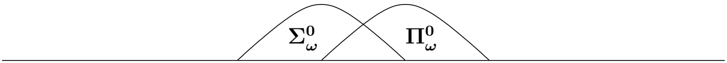


$\mathbf{\Sigma}^0_\omega$
$\mathbf{\Pi}^0_\omega$
$\bigcup_{n\in\mathbb{N}} \mathbf{\Delta}^0_n$
$\mathbf{\Delta}^0_5$
$\mathbf{\Delta}^0_4$
$\mathbf{\Delta}^0_3$
$BC(\mathbf{\Sigma}^0_2)$


$\mathbf{\Sigma^0_\omega}$
$\mathbf{\Pi^0_\omega}$
$\mathbf{\Delta^0_\omega} \neq \bigcup_{n\in\mathbb{N}} \mathbf{\Delta^0_n}$
$\mathbf{\Delta^0_5}$
$\mathbf{\Delta^0_4}$
$\mathbf{\Delta^0_3}$
$BC(\mathbf{\Sigma^0_2})$


$\boldsymbol{\Sigma}^0_\omega$
$\boldsymbol{\Delta}^0_\omega$
$\boldsymbol{\Pi}^0_\omega$
$\bigcup_{n\in\mathbb{N}} \boldsymbol{\Delta}^0_n$
$BC(\boldsymbol{\Sigma}^0_2)$


Σ^0_ω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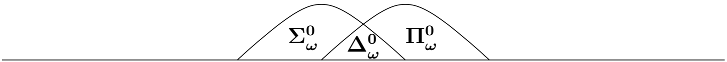
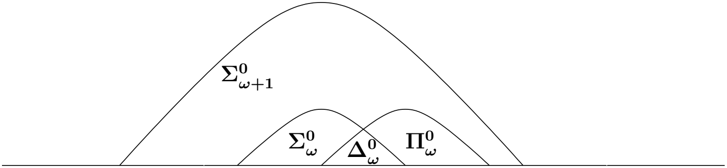
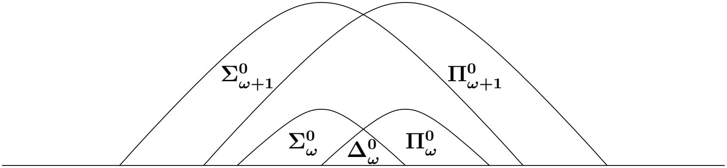
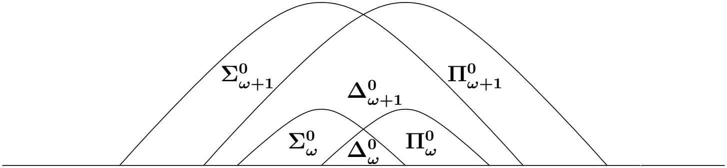
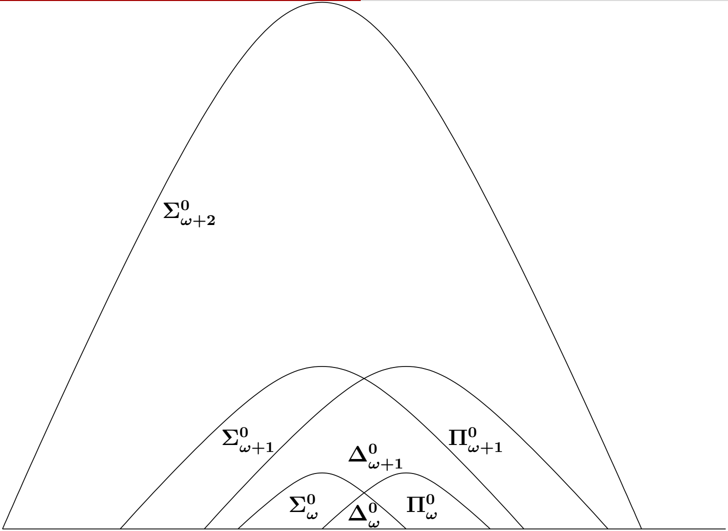
$\mathbf{\Pi}^0_\omega$


Σ^0_ω
Δ^0_ω
Π^0_ω


$\boldsymbol{\Sigma}^0_{\omega+1}$
$\boldsymbol{\Sigma}^0_{\omega}$
$\boldsymbol{\Delta}^0_{\omega}$
$\boldsymbol{\Pi}^0_{\omega}$


$\mathbf{\Sigma}^0_{\omega+1}$
$\mathbf{\Pi}^0_{\omega+1}$
$\mathbf{\Sigma}^0_{\omega}$
$\mathbf{\Delta}^0_{\omega}$
$\mathbf{\Pi}^0_{\omega}$


$\Sigma^0_{\omega+1}$
$\Pi^0_{\omega+1}$
$\Delta^0_{\omega+1}$
Σ^0_{ω}
Δ^0_{ω}
Π^0_{ω}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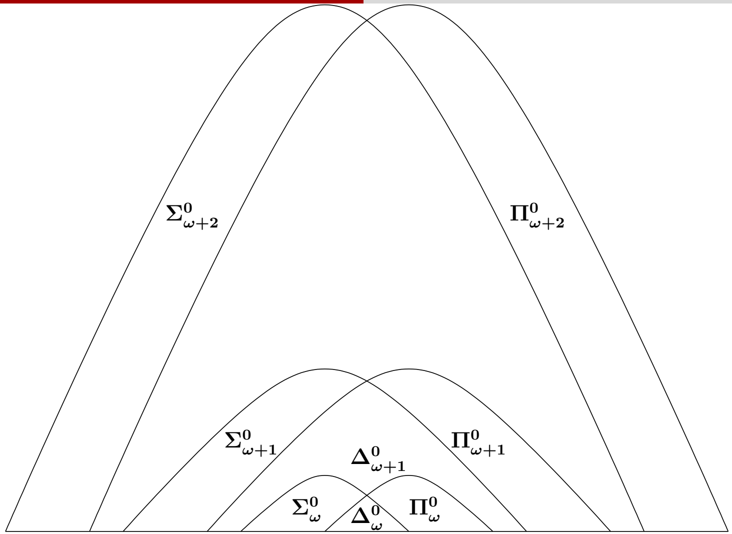
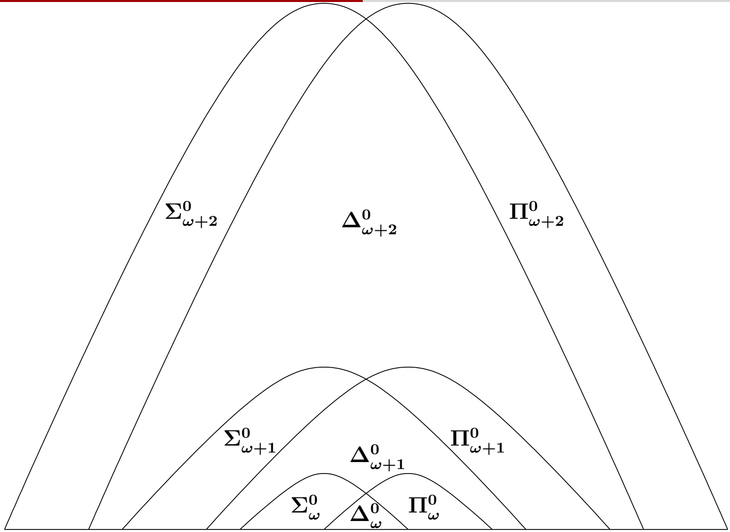
$\boldsymbol{\Sigma}^0_{\omega+1}$
$\boldsymbol{\Pi}^0_{\omega+1}$
$\boldsymbol{\Delta}^0_{\omega+1}$
$\boldsymbol{\Sigma}^0_{\omega}$
$\boldsymbol{\Pi}^0_{\omega}$
$\boldsymbol{\Delta}^0_{\omega}$


$\Sigma^0_{\omega+2}$
$\Pi^0_{\omega+2}$
$\Sigma^0_{\omega+1}$
$\Pi^0_{\omega+1}$
$\Delta^0_{\omega+1}$
Σ^0_{ω}
Δ^0_{ω}
Π^0_{ω}


$\boldsymbol{\Sigma}^0_{\omega+2}$
$\boldsymbol{\Delta}^0_{\omega+2}$
$\boldsymbol{\Pi}^0_{\omega+2}$
$\boldsymbol{\Sigma}^0_{\omega+1}$
$\boldsymbol{\Delta}^0_{\omega+1}$
$\boldsymbol{\Pi}^0_{\omega+1}$
$\boldsymbol{\Sigma}^0_{\omega}$
$\boldsymbol{\Delta}^0_{\omega}$
$\boldsymbol{\Pi}^0_{\omega}$

Borel Sets bis [1]

Example

Let $\mathscr{A} = (A, Q, q_i, \delta, F)$ be a *deterministic* Büchi automaton and $\vec{a}$ an infinite word,

$$\begin{aligned}\vec{a} \in \mathscr{L}(\mathscr{A}) \quad &\Longleftrightarrow \quad \text{there exists infinitely many } n \text{ s.t. } \rho_{\vec{a}}(n) \in F \\ &\Longleftrightarrow \quad \forall m \; \exists n > m \; \rho_{\vec{a}}(n) \in F \\ &\Longleftrightarrow \quad \forall m \; \exists n > m \; \rho_{\vec{a}} \in \mathscr{N}_n\end{aligned}$$

where

$$\mathscr{N}_n = \underbrace{\{\rho \in \{Q\}^\omega \mid \rho(n) \in F\}}_{open}$$

One notices that $f \; : \; \vec{a} \longrightarrow \rho_{\vec{a}}$ is continuous.

Borel Sets bis [2]

Example

$$\mathscr{O}_n = \{\vec{a} \in \{A\}^\omega \mid \rho_{\vec{a}} \in \mathscr{N}_n\} = \underbrace{f^{-1}\mathscr{N}_n}_{open}$$

Hence

$$\mathscr{L}(\mathscr{A}) = \underbrace{\bigcap_{m\in\mathbb{N}} \overbrace{\bigcup_{n>m} \mathscr{O}_n}^{\mathbf{\Sigma}^0_1}}_{\mathbf{\Pi}^0_2}$$

Borel Sets bis [3]

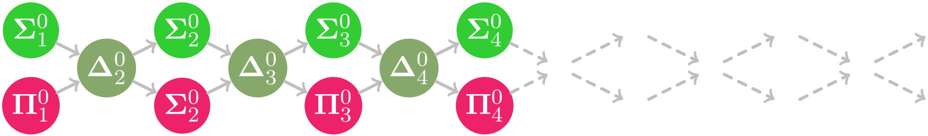


Figure: *Borel Hierarchy*

This is a characterization from below. Another one, from above relies on Suslin's theorem [6].

Borel Sets bis [4]

Definition (Analytic Set)

$\mathscr{A} \subseteq A^\omega$ is *analytic* if there exists some tree $T \subseteq (\mathbb{N} \times A)^*$ s.t.

$$\vec{a} \in \mathscr{A} \iff \exists \vec{x} \in \mathbb{N}^\omega \ \vec{x} \times \vec{a} \in [T].$$

where $\vec{x} \times \vec{a}$ stands for $(x_0, a_0)(x_1, a_1)(x_2, a_2)\ldots$

Same holds if one replaces $[T]$ by any Borel set.

Theorem (Suslin)

For all A countable and $B \subseteq A^\omega$,

$$B \ \mathit{Borel} \iff B \ \mathit{and} \ B^\complement \ \mathit{are\ both\ analytic}.$$

Example

Let $\mathscr{A} = (A, Q, q_i, \Delta, F)$ be any Büchi *non-deterministic* automaton, and $\vec{a}$ any infinite word,

$$\begin{aligned} \vec{a} \in \mathscr{L}(\mathscr{A}) \quad & \Longleftrightarrow \quad \text{there exists } \rho_{\vec{a}} \text{ and infinitely many } n \text{ s.t. } \rho_{\vec{a}}(n) \in F \\ & \Longleftrightarrow \quad \exists \rho_{\vec{a}}\ \forall m\ \exists n > m\ \rho_{\vec{a}}(n) \in F \\ & \Longleftrightarrow \quad \exists \rho \ \ (\rho, \vec{a}) \in \underbrace{\bigcap_m G_m}_{\mathbf{\Pi}^0_2} \end{aligned}$$

where

$$G_m = \underbrace{\{(\rho, \vec{a}) \in \mathbb{N}^\omega \times A^\omega \mid \big(\rho_m, a_m, \rho_{m+1}\big) \in \Delta \ \wedge \ \exists n > m \ \rho_n \in F\}}_{open}.$$

Borel Sets bis [6]

Example

- As the projection of a Borel set, this language is analytic.
- Since ω-regular languages are closed under complementation,

$$\mathscr{L}(\mathscr{A}) \text{ is Borel}$$

Theorem (Borel Determinacy, *Martin*)

Given any A and $B \subseteq A^\omega$ Borel,

the Gale-Stewart game $\mathscr{G}(B)$ is determined.

Reduction

Definition

- $X \ \leq \ Y \quad \Longleftrightarrow \quad \exists f$ *simple* $\ (x \in X \Leftrightarrow f(x) \in Y)$

- *simple* w.r. to topological complexity means **continuous**

Definition

A function $f : A^\omega \to B^\omega$ is **continuous** if for each open set $\mathscr{O} \subseteq B^\omega$, $f^{-1}\mathscr{O}$ is an open subset of A^ω .

- It corresponds to "*not lifting up the pen!*" on the real line.
- Here there is an elegant definition in terms of *games*.

Proposition

Soit $f : A^\omega \to B^\omega$,

$$f \text{ is continuous} \iff \text{player II has a w.s. in } \mathscr{C}(f).$$

Definition

Given $f : A^\omega \to B^\omega$, the game that characterizes continuous functions $\mathscr{C}(f)$ is an infinite game in which players (*I* and *II*) alternately chose $a \in A$ and $b \in B$. Player *I* starts. Player *II* can skip. Player *II* wins iff

$$f(\vec{a}) = \vec{b}.$$

Otherwise, *I* wins

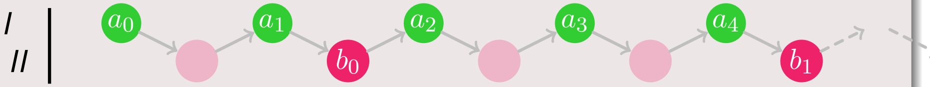


Figure: *Game* $\mathscr{C}(f)$.

Definition

- $X \leq Y \iff \exists f$ *simple* $(x \in X \Leftrightarrow f(x) \in Y)$
- Y is $\mathcal{C}-$**complete** $\iff \begin{cases} Y \in \mathcal{C} \\ X \leq Y \text{ , any } X \in \mathcal{C} \end{cases}$

- X is less complicated than Y

Reduction Games

Definition

- $X \leq_w Y \iff \exists f$ *continuous* $(x \in X \Leftrightarrow f(x) \in Y)$
  $\iff$ *II* **has a w.s. in** $\mathscr{W}(\mathbf{X}, \mathbf{Y})$

Wadge Ordering

Definition

- $X \;\leq_w\; Y \iff \exists f$ *continuous* $(x \in X \Leftrightarrow f(x) \in Y)$
  $\iff$ ***II* has a w.s. in** $\mathscr{W}(\mathbf{X}, \mathbf{Y})$

Continuous Reductions [6]

I $\mathscr{W}(X, Y)$ II

Continuous Reductions [7]

I $\mathscr{W}(X, Y)$ II

x_0

Continuous Reductions [8]

$$\textcolor{red}{\mathrm{I}} \quad \mathscr{W}\,(X, Y) \quad \textcolor{blue}{\mathrm{II}}$$

$\textcolor{red}{x_0}$

$\searrow$

Continuous Reductions [9]

$$\text{I} \quad \mathscr{W}(X,Y) \quad \text{II}$$

$$\begin{array}{ccc} x_0 & & \\ & \searrow & \\ & & s \end{array}$$

Continuous Reductions [10]

$$\begin{array}{ccc} \text{I} & \mathscr{W}(X,Y) & \text{II} \\ x_0 & & \\ & \searrow & \\ & & s \\ & \swarrow & \end{array}$$

Continuous Reductions [11]

$$\begin{array}{ccc}
\mathrm{I} & \mathscr{W}(X,Y) & \mathrm{II} \\
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & &
\end{array}$$

Continuous Reductions [12]

$$\begin{array}{ccc} \mathrm{I} & \mathscr{W}(X,Y) & \mathrm{II} \\ x_0 & & \\ & \searrow & \\ & & s \\ & \swarrow & \\ x_1 & & \\ & \searrow & \end{array}$$

Continuous Reductions [13]

$$
\begin{array}{ccc}
\text{I} & \mathscr{W}(X,Y) & \text{II} \\
x_0 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_1 & & \\
& \searrow & \\
& & s
\end{array}
$$

Continuous Reductions [14]

$$\begin{array}{ccc} \color{red}{\mathrm{I}} & \mathscr{W}(X,Y) & \color{blue}{\mathrm{II}} \\ \color{red}{x_0} & & \\ & \searrow & \\ & & \color{blue}{s} \\ & \swarrow & \\ \color{red}{x_1} & & \\ & \searrow & \\ & & \color{blue}{s} \\ & \swarrow & \end{array}$$

Continuous Reductions [15]

$$\begin{array}{ccc}
\mathrm{I} & \mathscr{W}(X,Y) & \mathrm{II} \\
x_0 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_1 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_2 & &
\end{array}$$

Continuous Reductions [16]

I	$\mathscr{W}(X,Y)$	II
x_0		
	$\searrow$	
		s
	$\swarrow$	
x_1		
	$\searrow$	
		s
	$\swarrow$	
x_2		
	$\searrow$	

Continuous Reductions [17]

$$\begin{array}{ccc}
\mathrm{I} & \mathscr{W}(X,Y) & \mathrm{II} \\
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_2 & & \\
 & \searrow & \\
 & & s
\end{array}$$

Continuous Reductions [18]

$$\begin{array}{ccc}
\mathbf{I} & \mathscr{W}(X,Y) & \mathbf{II} \\
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_2 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow &
\end{array}$$

Continuous Reductions [19]

$$\begin{array}{ccc}
\mathrm{I} & \mathscr{W}(X,Y) & \mathrm{II} \\
x_0 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_1 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_2 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_3 & &
\end{array}$$

Continuous Reductions [20]

$$\begin{array}{ccc}
\mathrm{I} & \mathscr{W}(X,Y) & \mathrm{II} \\
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_2 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_3 & & \\
 & \searrow &
\end{array}$$

Continuous Reductions [21]

$$\begin{array}{ccc}
\mathrm{I} & \mathscr{W}(X,Y) & \mathrm{II} \\
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_2 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_3 & & \\
 & \searrow & \\
 & & y_1
\end{array}$$

Continuous Reductions [22]

$$\begin{array}{ccc}
\color{red}{\mathrm{I}} & \mathscr{W}(X,Y) & \color{blue}{\mathrm{II}} \\
\color{red}{x_0} & & \\
& \searrow & \\
& & \color{blue}{s} \\
& \swarrow & \\
\color{red}{x_1} & & \\
& \searrow & \\
& & \color{blue}{s} \\
& \swarrow & \\
\color{red}{x_2} & & \\
& \searrow & \\
& & \color{blue}{s} \\
& \swarrow & \\
\color{red}{x_3} & & \\
& \searrow & \\
& & \color{blue}{y_1} \\
& \swarrow &
\end{array}$$

Continuous Reductions [23]

$$
\begin{array}{ccc}
\mathrm{I} & \mathscr{W}(X,Y) & \mathrm{II} \\
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_2 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_3 & & \\
 & \searrow & \\
 & & y_1 \\
 & \swarrow & \\
x_4 & &
\end{array}
$$

Continuous Reductions [24]

I	$\mathscr{W}(X,Y)$	II
x_0		
	$\searrow$	
		s
	$\swarrow$	
x_1		
	$\searrow$	
		s
	$\swarrow$	
x_2		
	$\searrow$	
		s
	$\swarrow$	
x_3		
	$\searrow$	
		y_1
	$\swarrow$	
x_4		
	$\searrow$	

Continuous Reductions [25]

$$
\begin{array}{ccc}
\color{red}{\mathsf{I}} & \mathscr{W}(X,Y) & \color{blue}{\mathsf{II}} \\
\color{red}{x_0} & & \\
 & \searrow & \\
 & & \color{blue}{s} \\
 & \swarrow & \\
\color{red}{x_1} & & \\
 & \searrow & \\
 & & \color{blue}{s} \\
 & \swarrow & \\
\color{red}{x_2} & & \\
 & \searrow & \\
 & & \color{blue}{s} \\
 & \swarrow & \\
\color{red}{x_3} & & \\
 & \searrow & \\
 & & \color{blue}{y_1} \\
 & \swarrow & \\
\color{red}{x_4} & & \\
 & \searrow & \\
 & & \color{blue}{y_2}
\end{array}
$$

Continuous Reductions [26]

$$\begin{array}{ccc}
\mathrm{I} & \mathscr{W}(X,Y) & \mathrm{II} \\
x_0 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_1 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_2 & & \\
& \searrow & \\
& & s \\
& \swarrow & \\
x_3 & & \\
& \searrow & \\
& & y_1 \\
& \swarrow & \\
x_4 & & \\
& \searrow & \\
& & y_2 \\
& \swarrow &
\end{array}$$

Continuous Reductions [27]

$$\begin{array}{ccc}
\textcolor{red}{\mathrm{I}} & \mathscr{W}(X,Y) & \textcolor{blue}{\mathrm{II}} \\
x_0 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_1 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_2 & & \\
 & \searrow & \\
 & & s \\
 & \swarrow & \\
x_3 & & \\
 & \searrow & \\
 & & y_1 \\
 & \swarrow & \\
x_4 & & \\
 & \searrow & \\
 & & y_2 \\
 & \swarrow & \\
\hline
x & & y
\end{array}$$

II wins iff
$(x \in X \leftrightarrow y \in Y)$

Wadge Ordering

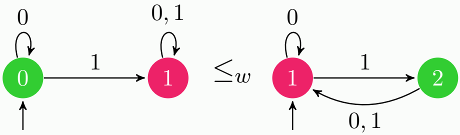
Definition

- $X \;\leq_w\; Y \iff \exists f$ *continuous* $(x \in X \Leftrightarrow f(x) \in Y)$
  $\iff$ *II* **has a w.s. in** $\mathscr{W}(\mathbf{X}, \mathbf{Y})$
- Y is $\mathcal{C}-$**complete** $\iff \begin{cases} Y \in \mathcal{C} \\ X \;\leq_w\; Y \;, \text{ any } X \in \mathcal{C} \end{cases}$
- $\mathcal{C}$ is a **Wadge Class** $\iff$ some $Y \in \mathcal{C}$ is $\mathcal{C}-$**complete**

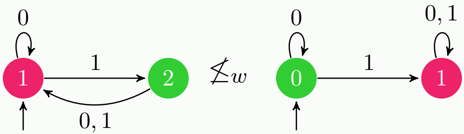
- $L <_w M$ stands for $L \leq_w M$ and $M \not\leq_w L$.
- $L \equiv_w M$ stands for $L \leq_w M$ and $M \leq_w L$.
- $X \;\leq_w\; Y \iff X^{\complement} \;\leq_w\; Y^{\complement}$

Continuous Reductions [29]

Example

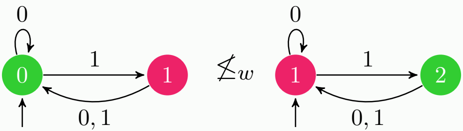


Example

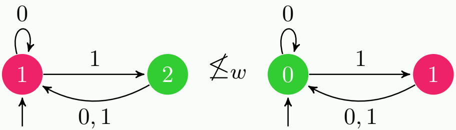


Continuous Reductions [30]

Example



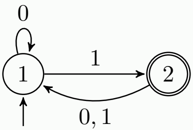
Example



Continuous Reductions [31]

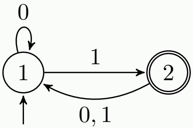
Example

The following Büchi automaton $\mathscr{B}$ is $\mathbf{\Pi}^0_2$-complete.



- Since $\mathscr{L}(\mathscr{B})$ is deterministic Büchi, $\mathscr{L}(\mathscr{B}) \in \mathbf{\Pi}^0_2$
- Let $B = \bigcap_{n\in\mathbb{N}} \mathscr{O}_n$ be any $\mathbf{\Pi}^0_2$-subset of A^ω. We show

$$B = \bigcap_{n\in\mathbb{N}} \mathscr{O}_n \quad \leq_w$$



The relation $\leq_w$ is a partial ordering:

- reflexive
- transitive

With determinacy:

1. anti-chaines have length at most two;
2. it is well-founded, i.e. there is no infinite descending chain

$$A_0 >_w A_1 >_w A_2 >_w \ldots >_w A_n >_w A_{n+1} >_w \ldots\ldots$$

Continuous Reductions [33]

1. The first result is an immediate consequence of the following lemma

Lemma (Wadge)

Given $L \subseteq A^\omega$ and $M \subseteq B^\omega$, if $\mathscr{W}(L, M)$ is determined, then

$$L \not\leq_w M \Longrightarrow M \leq_w L^\complement.$$

2. The second relies on an elegant construction [6].

Continuous Reductions [34]

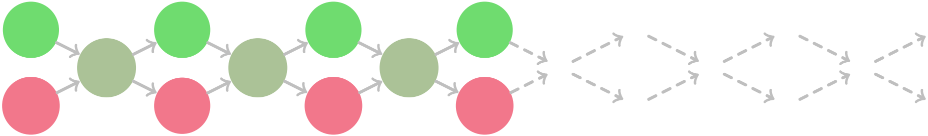


Figure: *The Wadge Hierarchy.*

Proposition

If $\mathscr{A}$ is some deterministic parity automaton, then

$$\mathscr{L}(\mathscr{A}) \in \mathbf{\Delta}_3^0 = \mathbf{\Sigma}_3^0 \cap \mathbf{\Pi}_3^0.$$

Continuous Reductions [35]

$$
\begin{aligned}
&\vec{a} \in \mathscr{L}(\mathscr{A}) \\
&\iff \exists i \leq p \left(\exists^{\infty} n\ \mathcal{C}\left(\rho_{\vec{a}}(n)\right) = 2i\ \wedge\ \exists m\ \forall n \geq m\ \mathcal{C}\left(\rho_{\vec{a}}(n)\right) \leq 2i\right) \\
&\iff \underbrace{\exists i \leq p \left(\underbrace{\forall m\ \exists n > m\ \mathcal{C}\left(\rho_{\vec{a}}(n)\right) = 2i}_{\mathbf{\Pi}^0_2}\ \wedge\ \underbrace{\exists m\ \forall n \geq m\ \mathcal{C}\left(\rho_{\vec{a}}(n)\right) \leq 2i}_{\mathbf{\Sigma}^0_2}\right)}_{\mathbf{\Sigma}^0_3}.
\end{aligned}
$$

Since $\mathscr{L}(\mathscr{A})^{\complement} = \mathscr{L}(\mathscr{A}^{\complement})$, we get $\mathscr{L}(\mathscr{A}) \in \mathbf{\Pi}^0_3$.

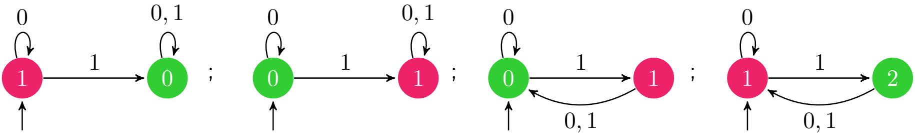


Figure: *ω-regular languages complete resp. for* $\boldsymbol{\Sigma}^0_1$, $\boldsymbol{\Pi}^0_1$, $\boldsymbol{\Sigma}^0_2$, $\boldsymbol{\Pi}^0_2$.

- As shown by Wagner [12] and Selivanov [11], the Wadge ordering yields a much finer analysis.
- For this purpose, we consider (The following *well-ordering*)

  the set of all *finite decreasing (at large) sequences of integers*

  equipped with the *lexicographic ordering* $\leq_{lex}$

  e.g.

$$88888764443322222222222221000$$

Continuous Reductions [37]

- This well-ordering is isomorphic to the ordinal ω^ω.
  The isomorphism maps

$$n_k \geq n_{k-1} \geq \ldots \geq n_0 \text{ to } \omega^{n_k} + \omega^{n_{k-1}} + \ldots + \omega^{n_0}.$$

- To each such finite sequence u we associate a deterministic parity automaton $\mathscr{A}_u$ s.t

$$u <_{lex} v \iff \mathscr{A}_u <_w \mathscr{A}_v.$$

- We first define for each integer n $\mathscr{A}_n$

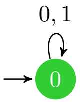


Figure: *Automaton $\mathscr{A}_0$.*

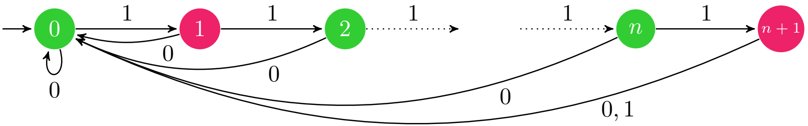


Figure: *Automaton $\mathscr{A}_{n+1}$ (the coloring corresponds to n even).*

- To each sequence $u = n_k n_{k-1} \dots n_0$ satisfying $n_k \geq n_{k-1} \geq \dots \geq n_0$, we associate three automata

1. $\mathscr{A}_u$,
2. $-\mathscr{A}_u$,
3. $\pm\mathscr{A}_u$

whose graph are represented by the following figures.

  - the labelling does not matter as long as it makes them deterministic.

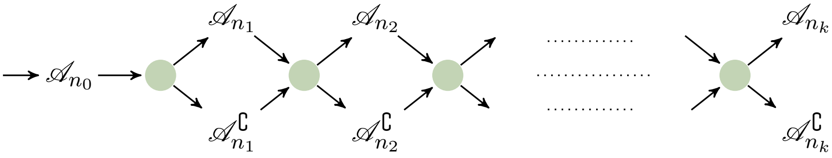


Figure: *Automaton* $\mathscr{A}_{n_k \dots n_0}$.

Continuous Reductions [40]

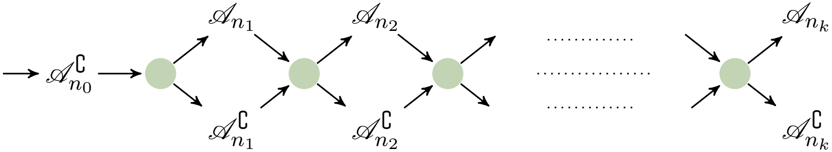


Figure: *Automaton* $-\mathscr{A}_{n_k \dots n_0}$.

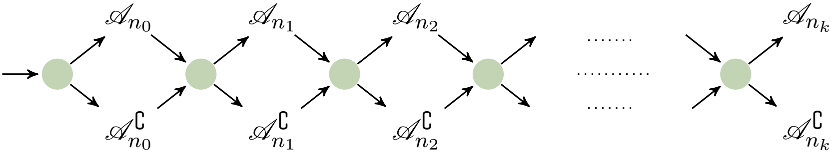


Figure: *Automaton* $\pm\mathscr{A}_{n_k \dots n_0}$.

Example

A deterministic labelling for $\mathscr{A}_{7100}$.

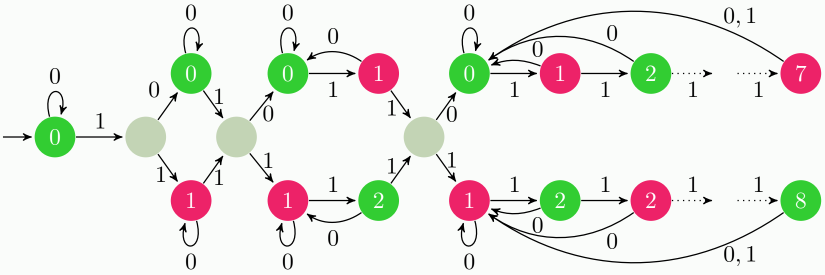


Figure: *Automaton* $\mathscr{A}_{7100}$.

Continuous Reductions [42]

Proposition

Given u, v two finite decreasing sequences of integers.

1. $\mathscr{A}_u \not\leq_w -\mathscr{A}_u$ *and* $-\mathscr{A}_u \not\leq_w \mathscr{A}_u$
2. $\mathscr{A}_u <_w \pm\mathscr{A}_u$ *and* $-\mathscr{A}_u <_w \pm\mathscr{A}_u$
3. *If* $u <_{lex} v$, *then* $\pm\mathscr{A}_u <_w \mathscr{A}_v$ *and* $\pm\mathscr{A}_u <_w -\mathscr{A}_v$.

Continuous Reductions [43]

Theorem

If $\mathscr{A}$ is any deterministic parity automaton, then there exists some non-empty finite decreasing sequence of integers u s.t. (only) one of the following three possibilitis occurs:

1. $\mathscr{A} \equiv_w \mathscr{A}_u$
2. $\mathscr{A} \equiv_w -\mathscr{A}_u$
3. $\mathscr{A} \equiv_w \pm\mathscr{A}_u$.

Counter Automata and Tree Automata [1]

- If one considers *PushDown automata*, or even *1-counter automata* (with Büchi acceptance conditions)

  The Wadge hierarchy of languages recognized by non-deterministic such machines is **inextricable**[3].

  - *Olivier Finkel* showed that it is as complicated as the same problem for *Turing machines*

Counter Automata and Tree Automata [2]

- If one considers *infinite-tree-automata*,
  - In case of *deterministic parity automata*, *Damian Niwiński* and *Igor Walukiewicz* [9] showed that the languages recognized are either complete for the class of co-analytic sets, or they sit inside the class $\mathbf{\Pi}^0_3$.
  - Later *Filip Murlak* gave a complete description of its Wadge hierarchy [8].
- In case of *non-deterministic parity automata*, the Wadge hierarchy of ω-regular tree languages still highly remains a mystery.